One-Bend Drawings of Outerplanar Graphs Inside Simple Polygons

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Graph Drawing 2021

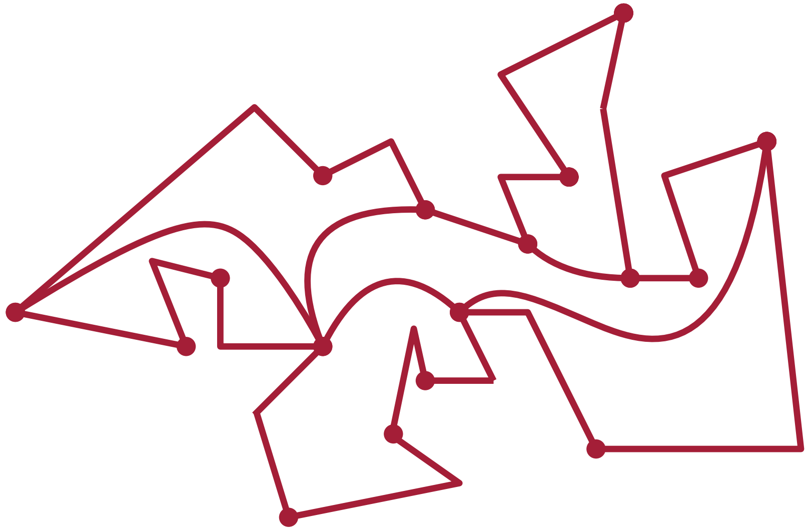
Problem

Input

- Biconnected outerplanar graph $G = (V, E)$
- Drawing of the outer face of G as simple polygon P

Question

Does G admit an outerplanar drawing with at most one bend per edge that contains the given drawing of the outer face?



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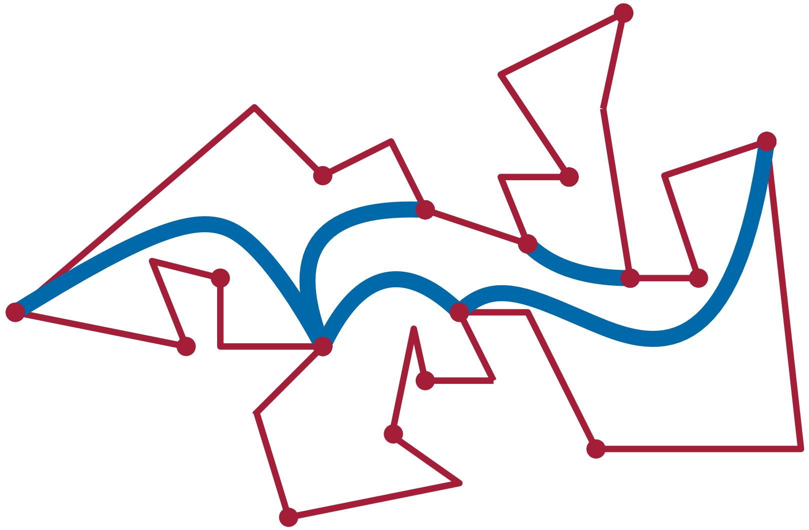

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Motivation & Related Work

- Prescribing the outer face: many results

[Tutte '63] [Hong, Nagamochi '07]

- Extension of straight-line drawings: NP-hard

[Patrignani '06]

- Allowing overlaps of edges and polygonal region: $\exists\mathbb{R}$-complete

[Lubiw, Miltzow, Mondal '18]

Non-straight edges:

- Extension of planar topological drawing: linear time

[Angelini et al '15]

- Linear number of bends per edge suffices (72 $|P|$)

[Chan et al. '15]

- P star-shaped: one bend suffices

[Mchedlize, Urhausen '18]

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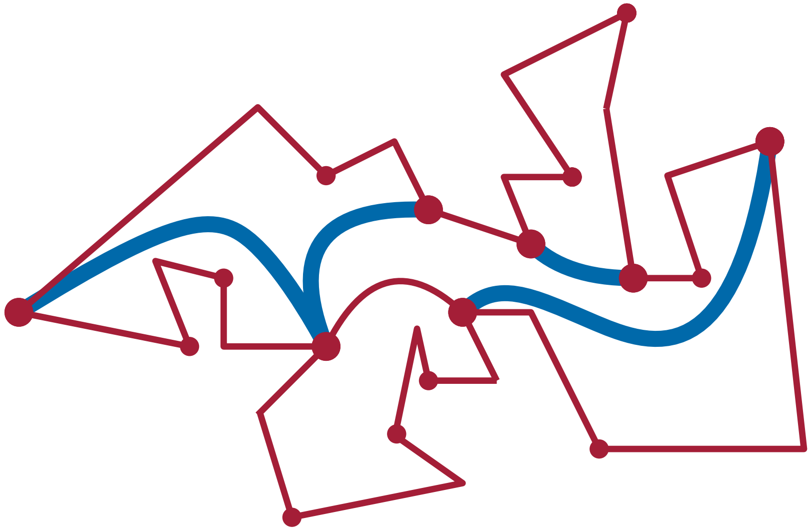

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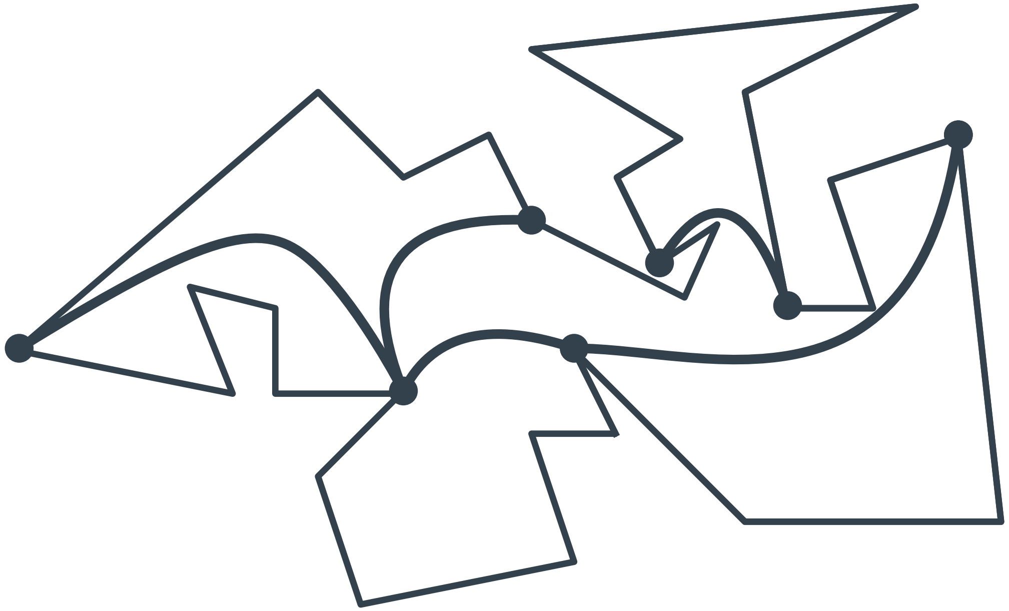
G admits an outerplanar drawing with at most one bend per edge that contains the given drawing of the outer face?

Can be decided in $O(nm)$ time

(n: # vertices, m: # interior edges).

Algorithm overview

- G is a biconnected outerplanar graph ⇒ use dual tree
- Traverse the tree bottom-up, incrementally process edges and refine the polygon P



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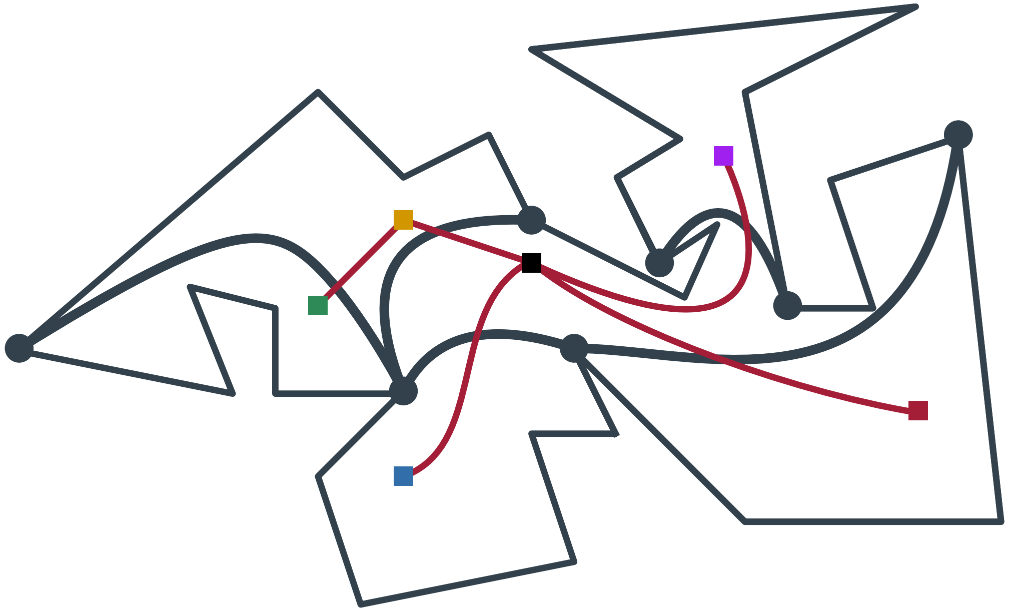

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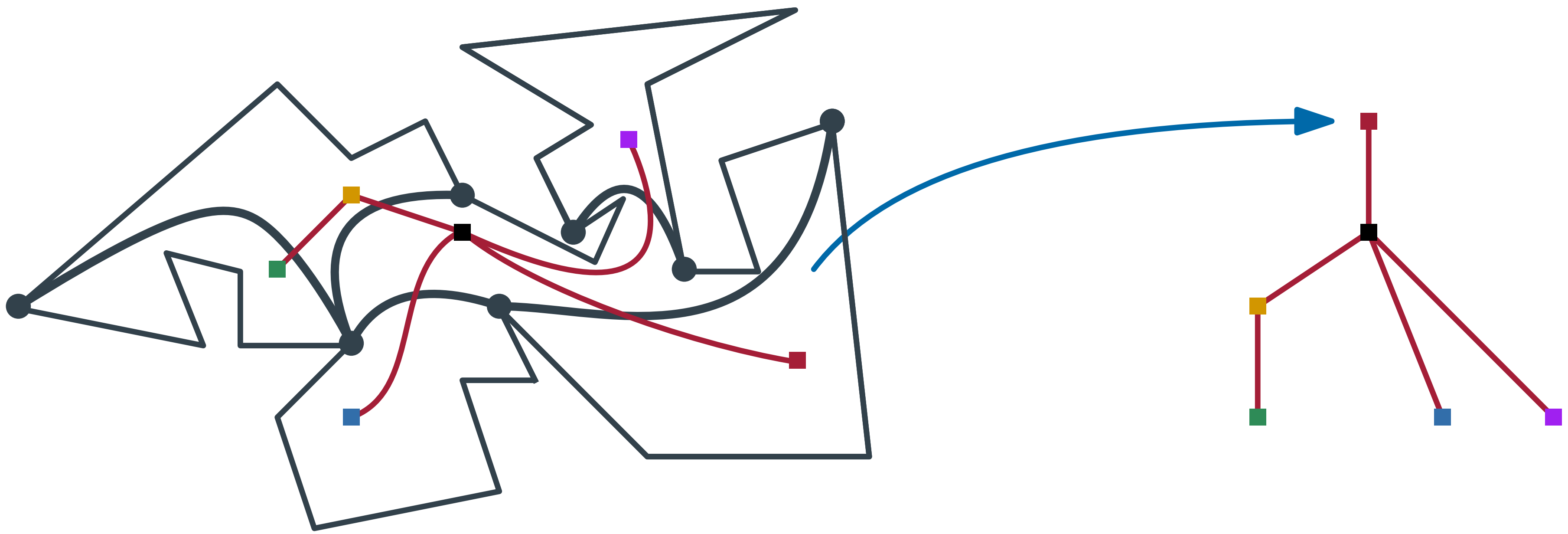

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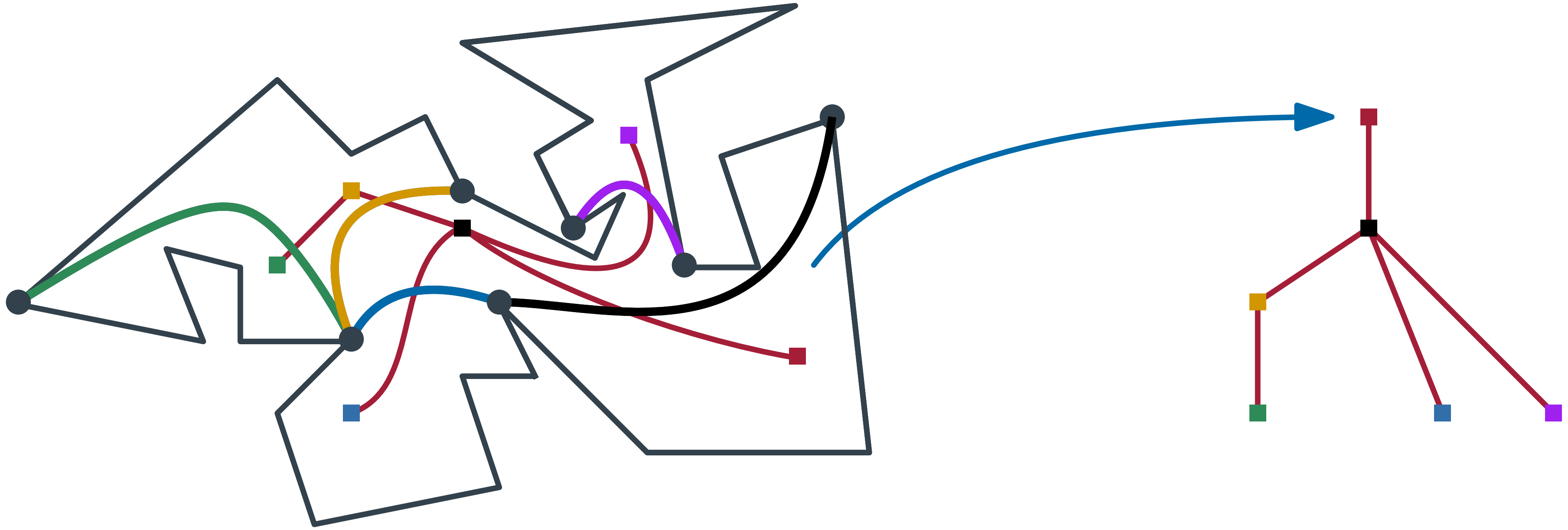
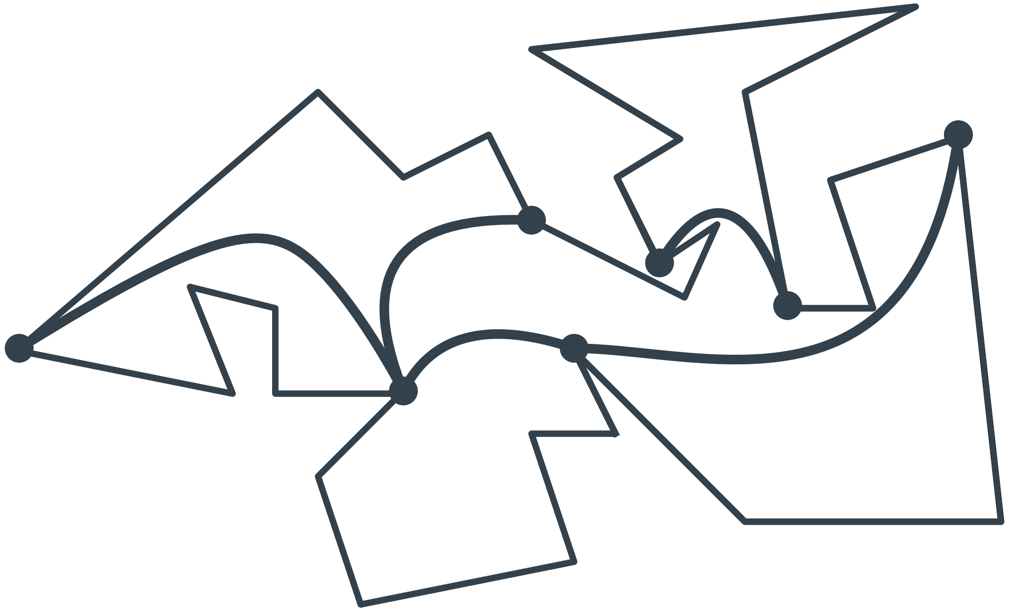
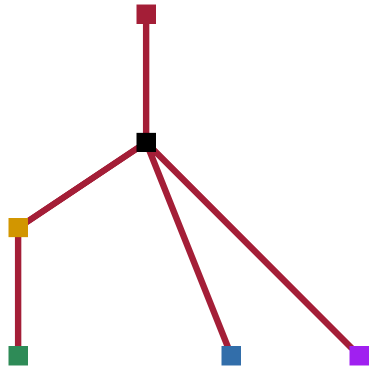


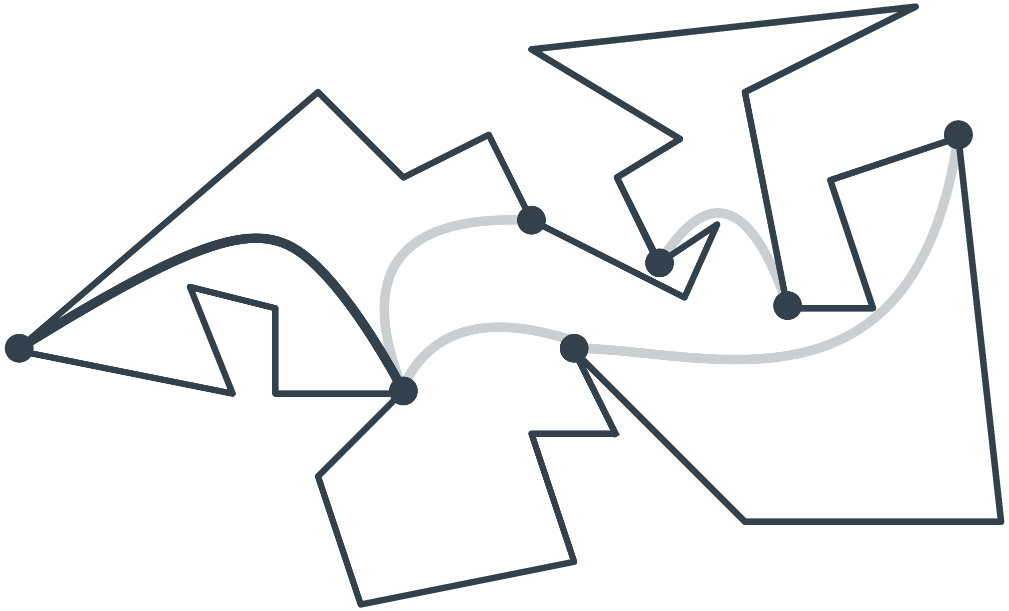
Algorithm overview

- G is a biconnected outerplanar graph ⇒ use dual tree
- Traverse the tree bottom-up, incrementally process edges and refine the polygon P
- Graph fits in polygon ⇔ sub-graph fits in refined polygon
- Place the bend points for the edges in a top-down traversal





Details: Refine the polygon



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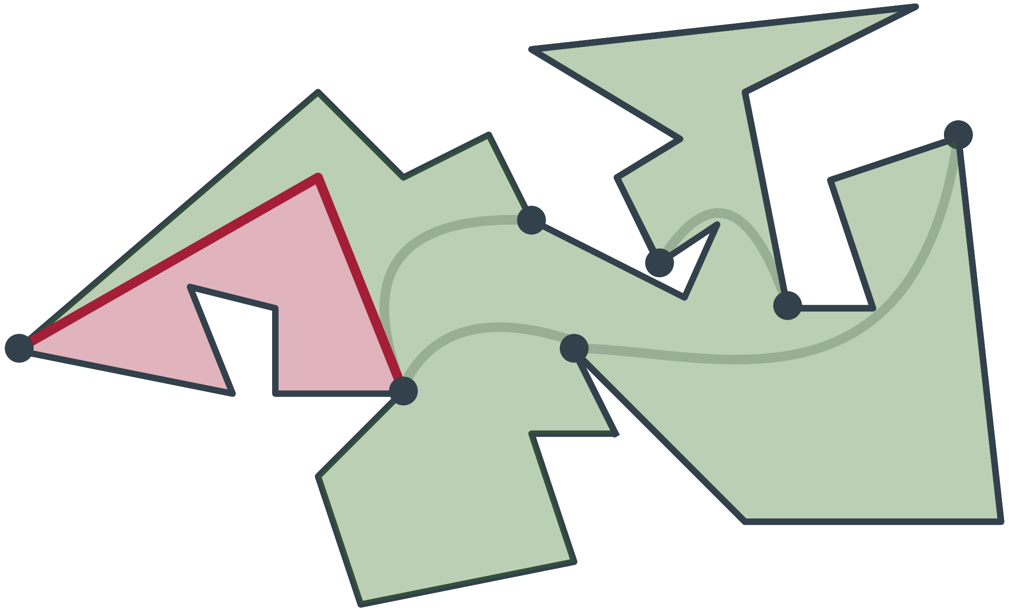

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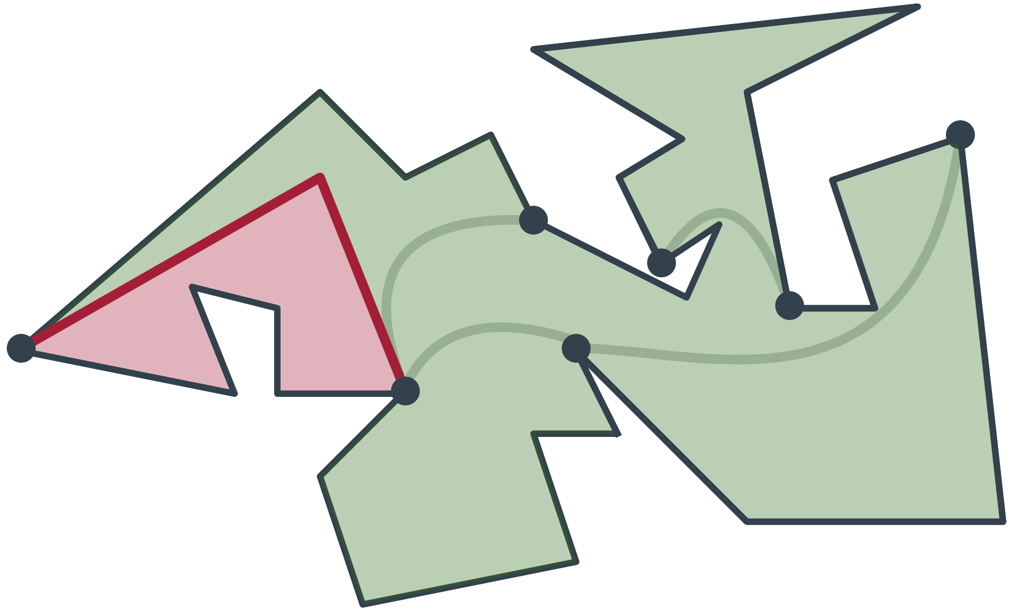

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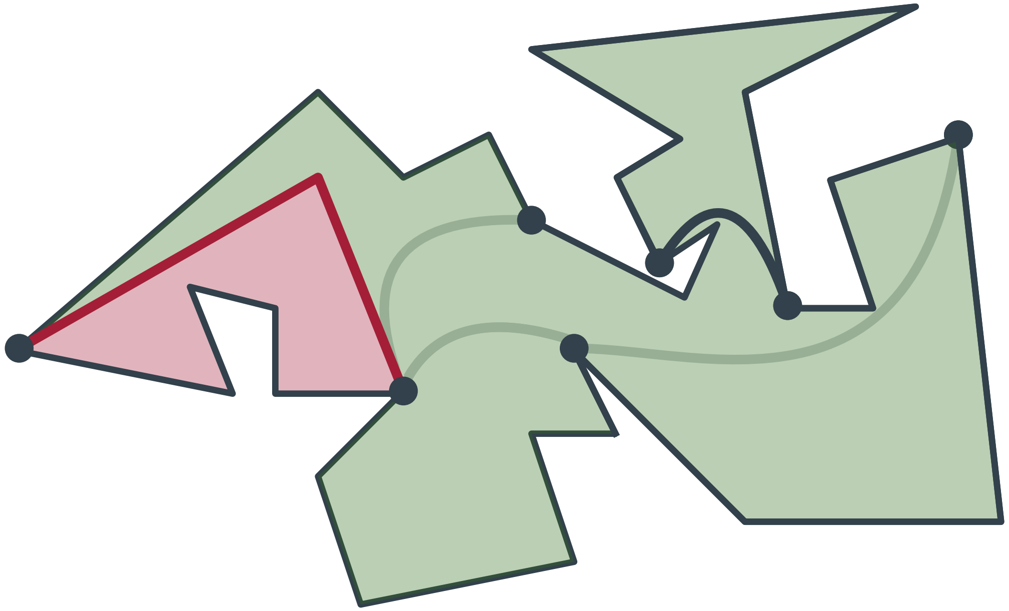

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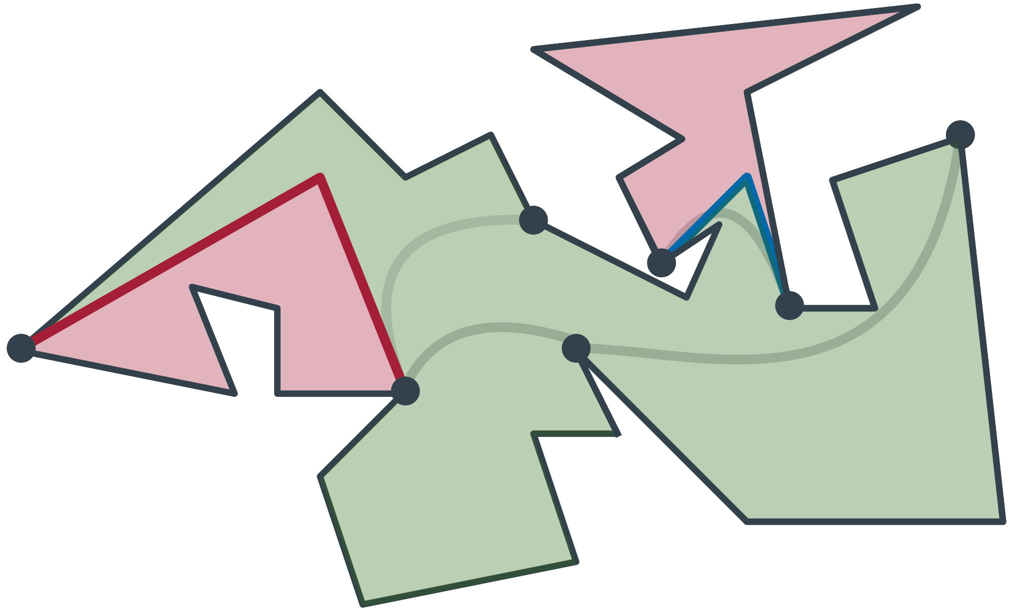

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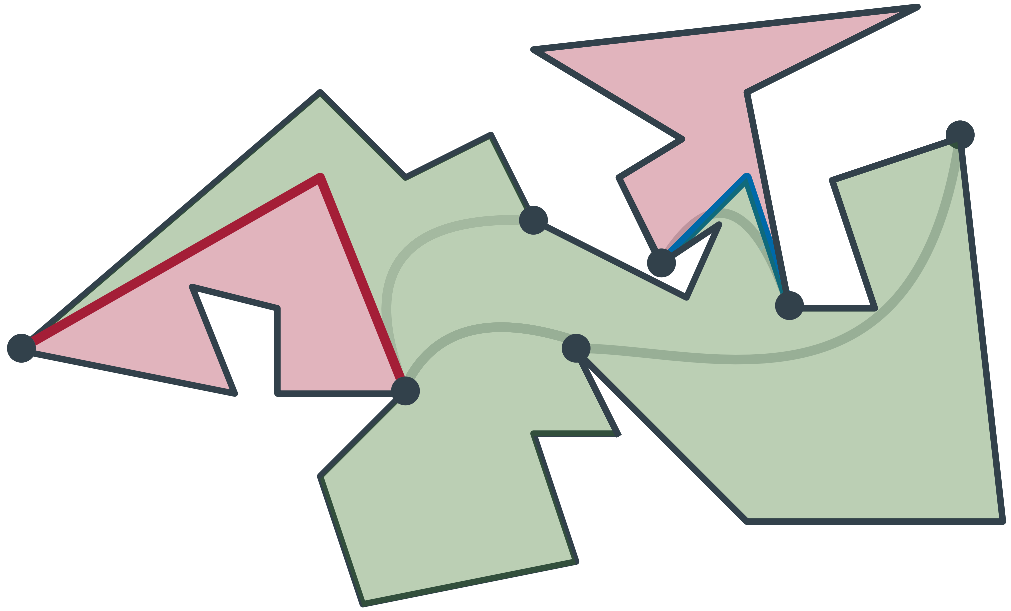

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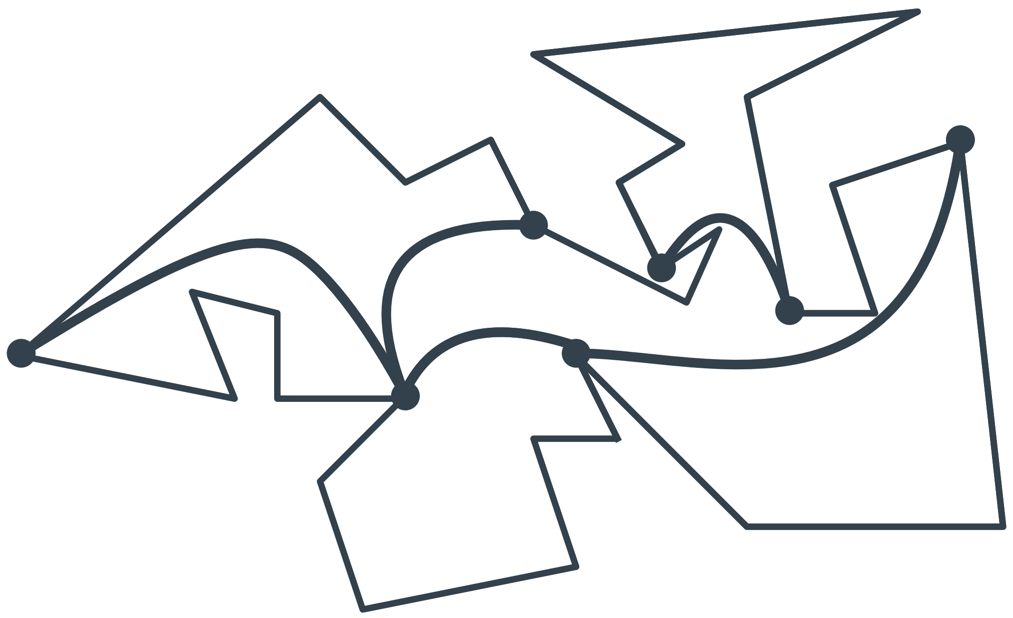

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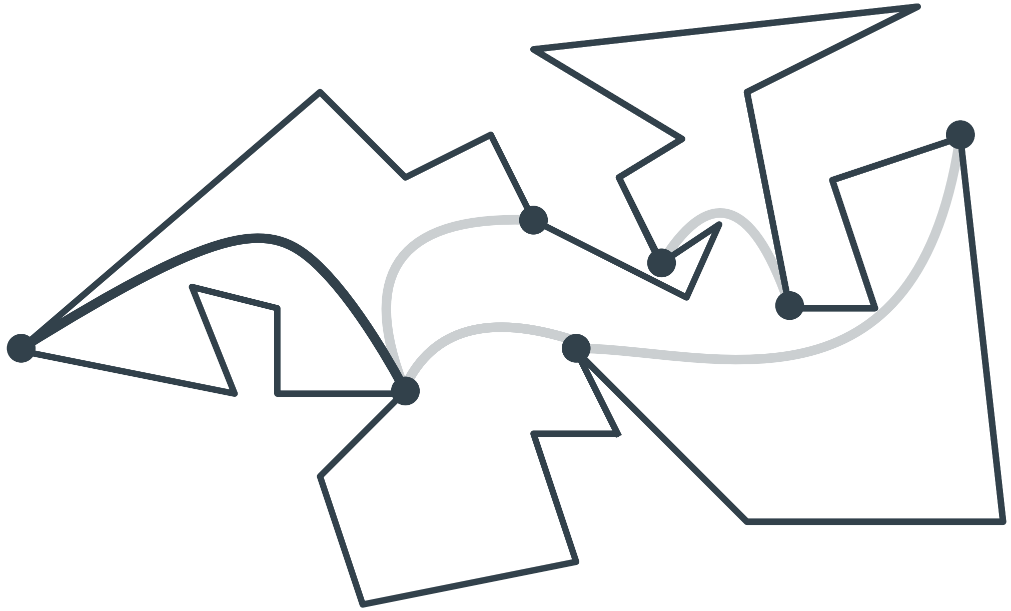

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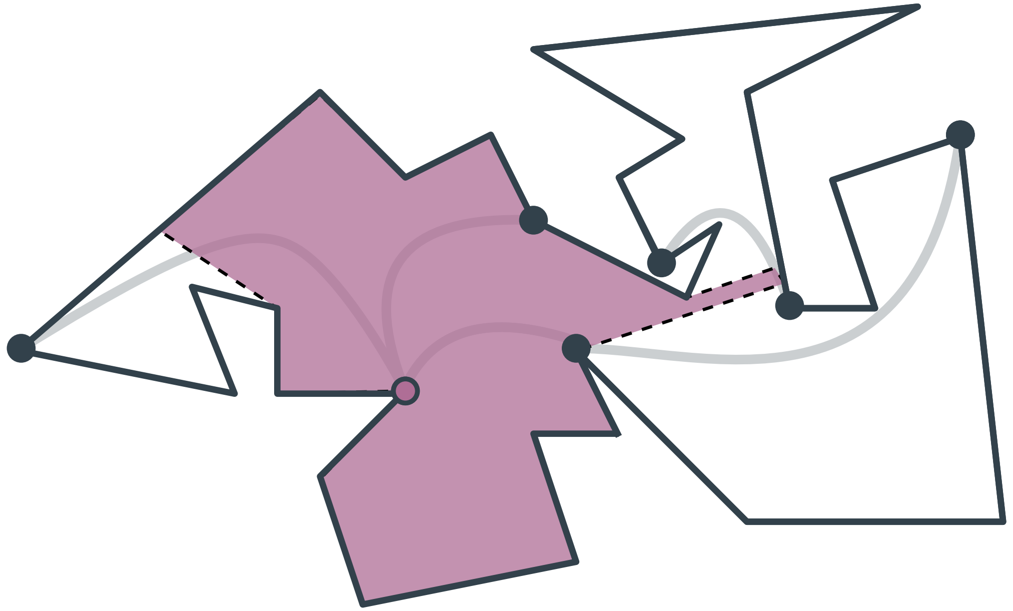

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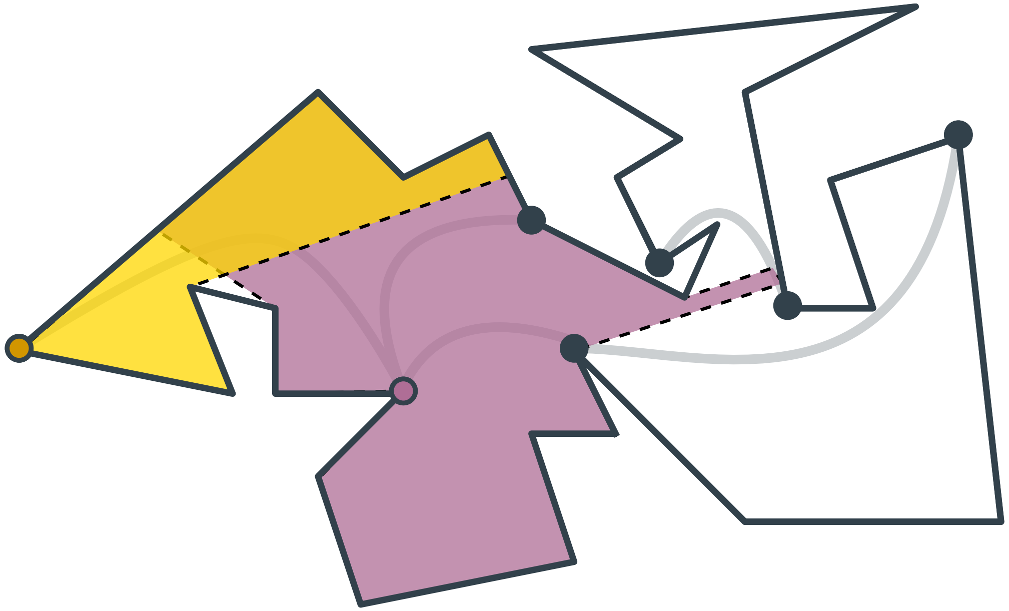

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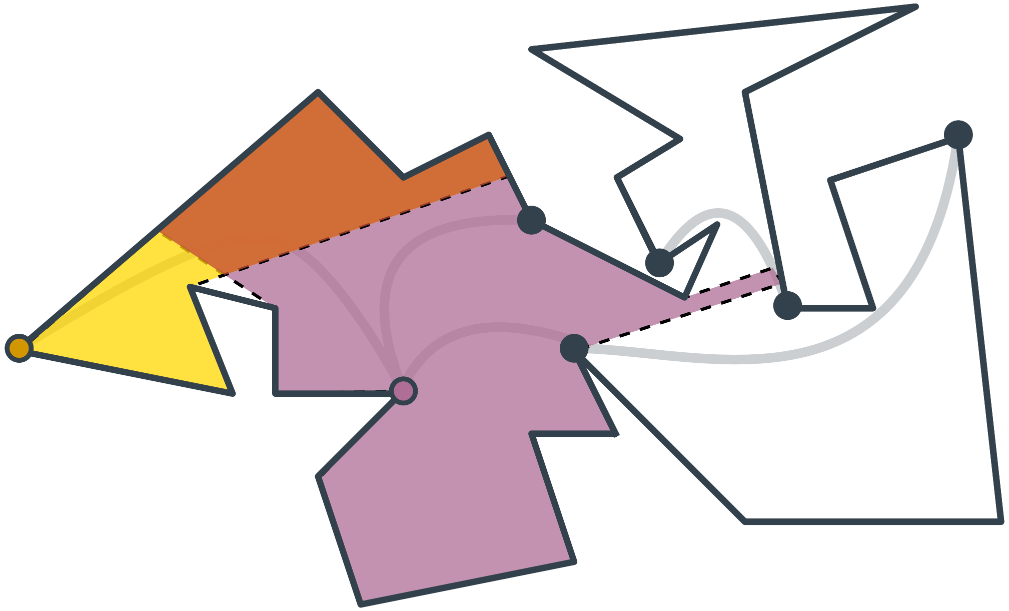

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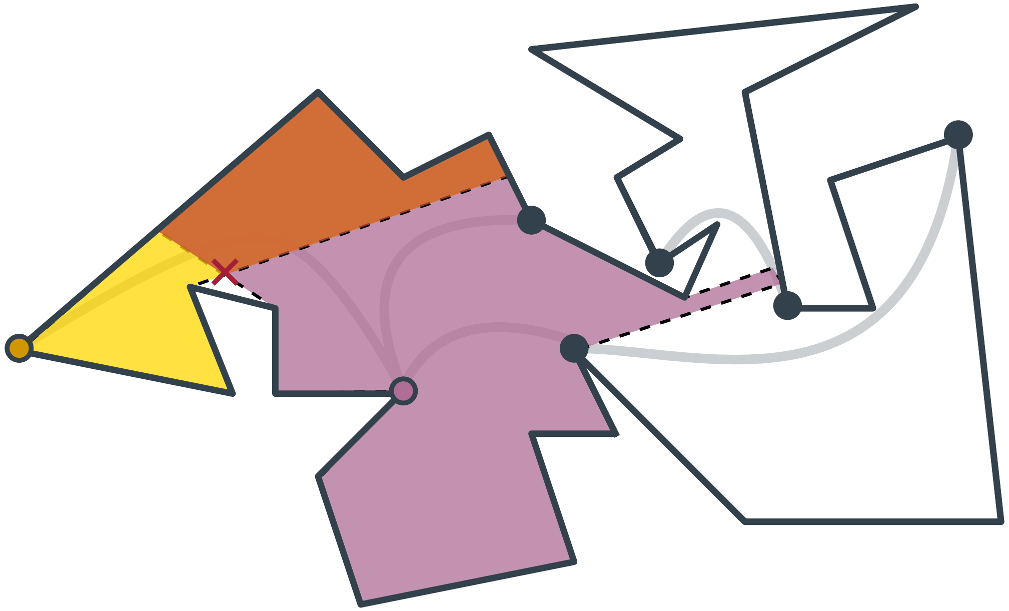

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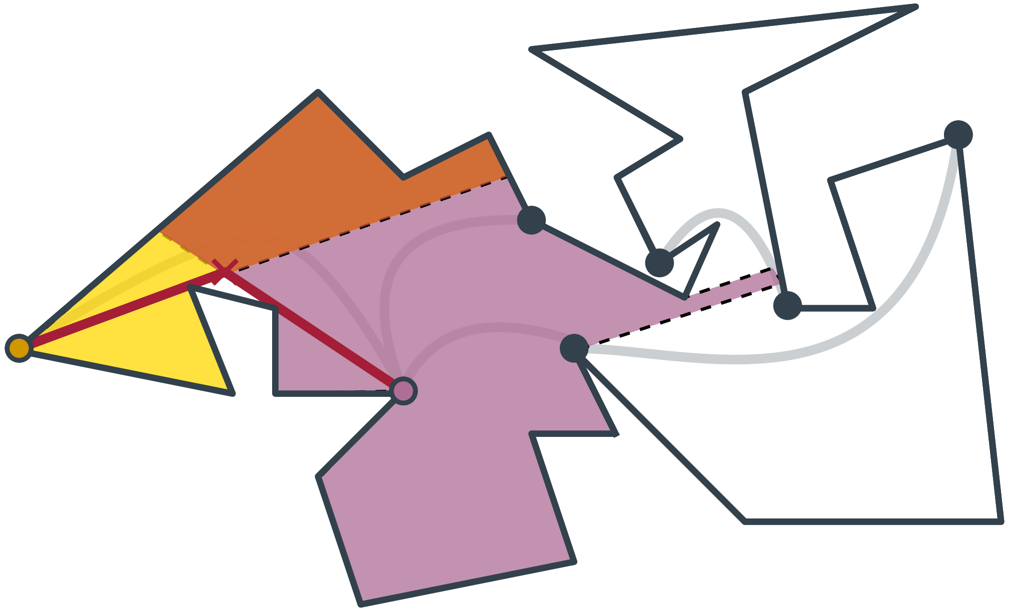

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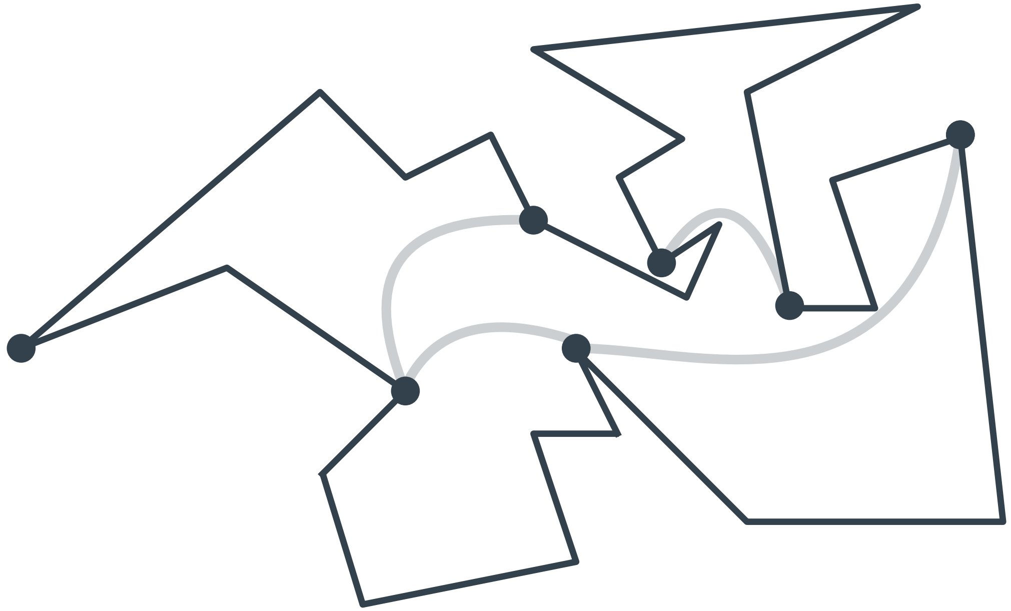

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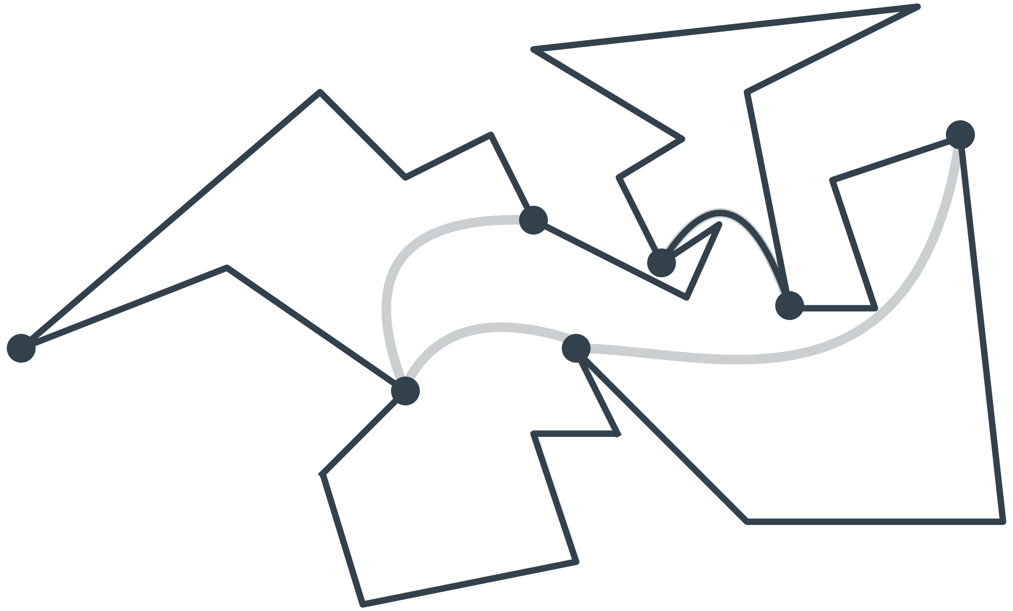

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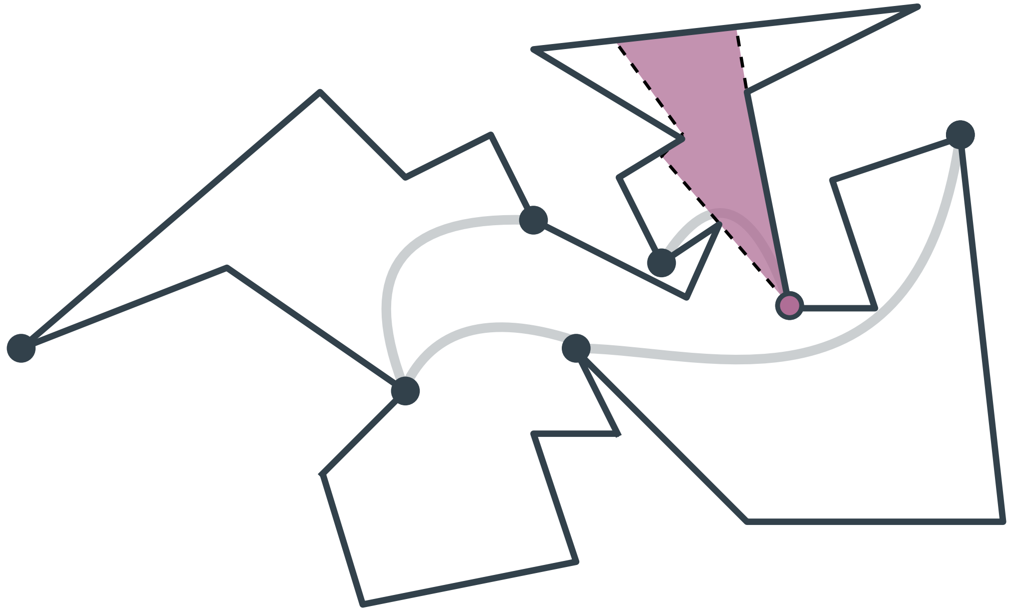

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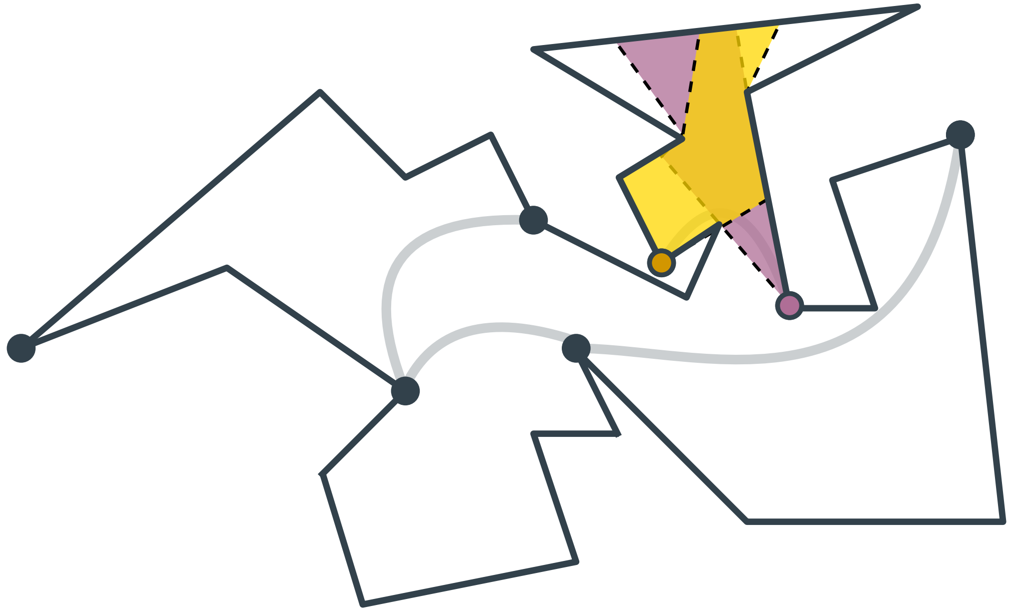

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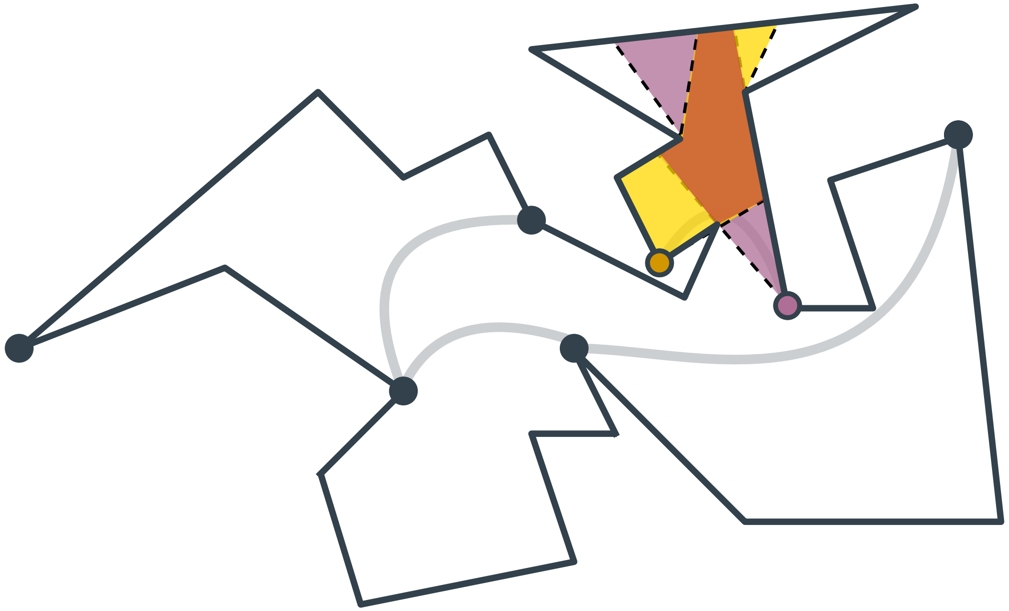

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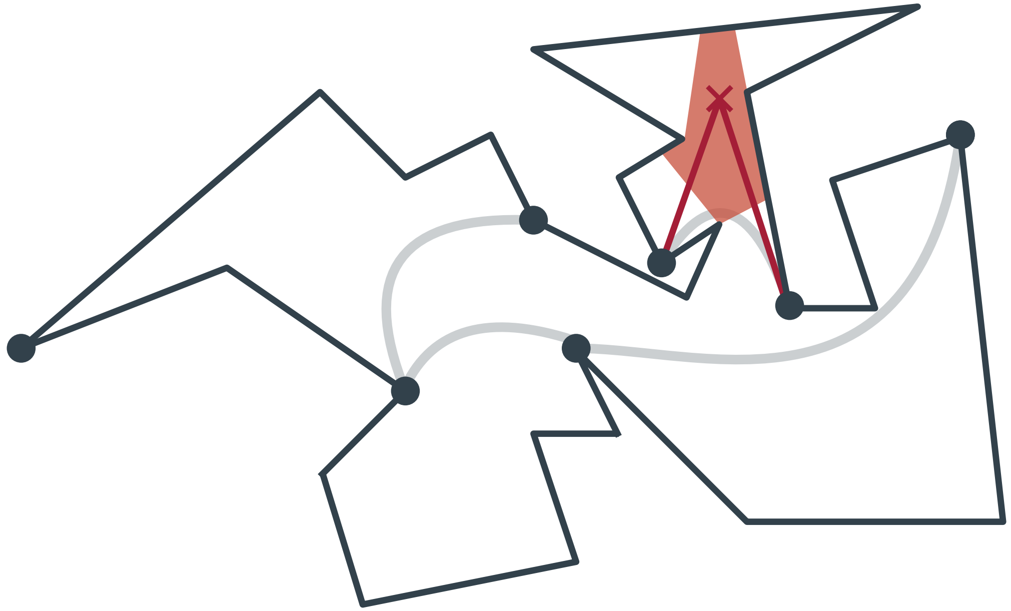

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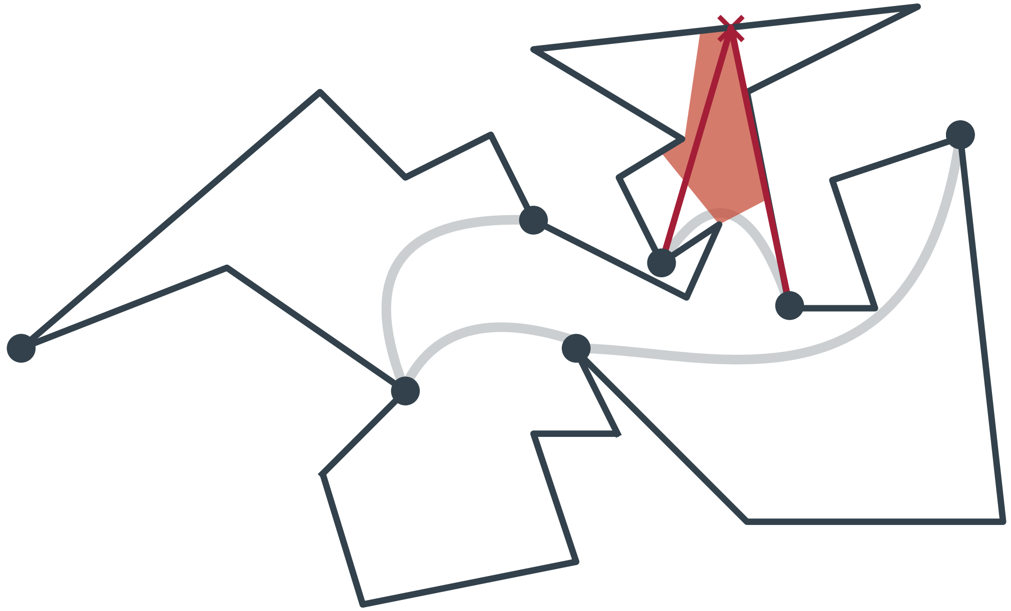

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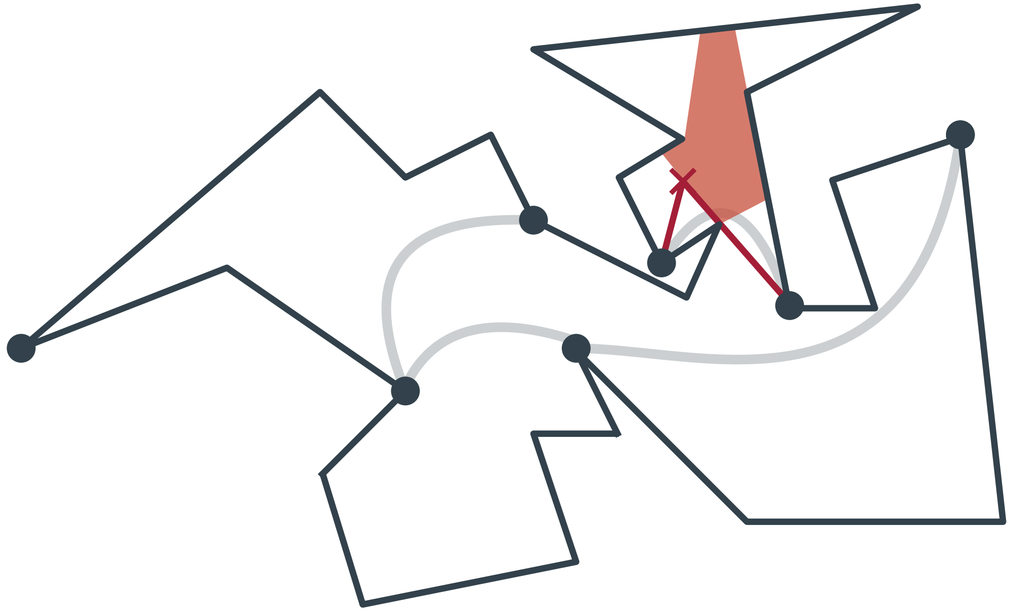

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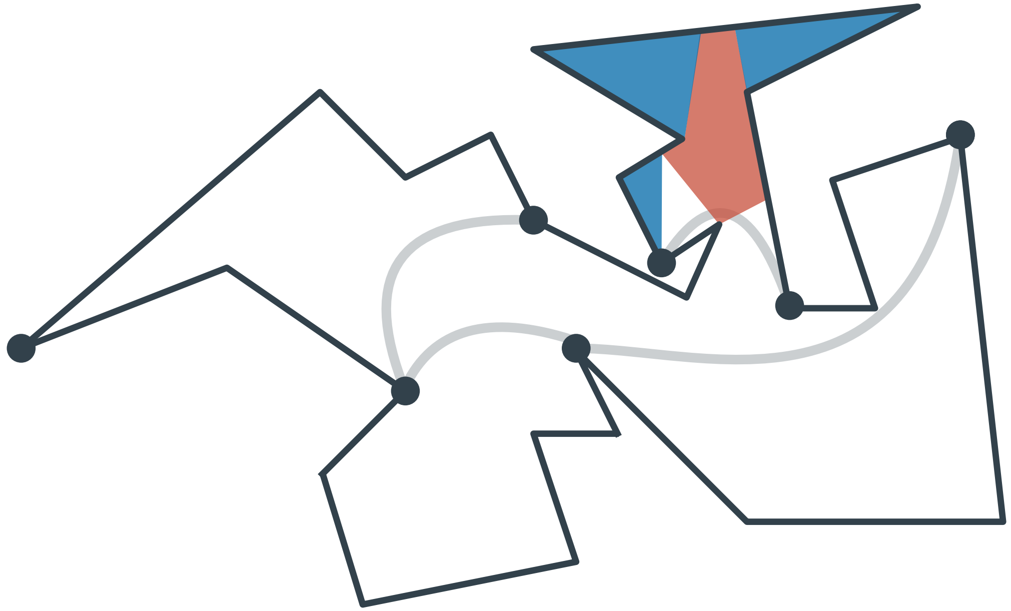

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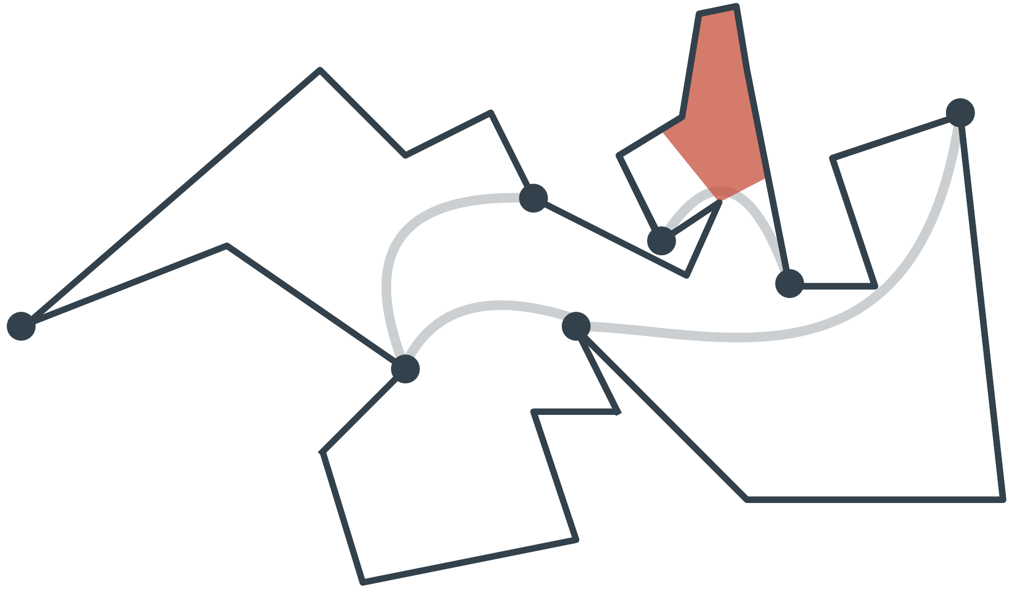

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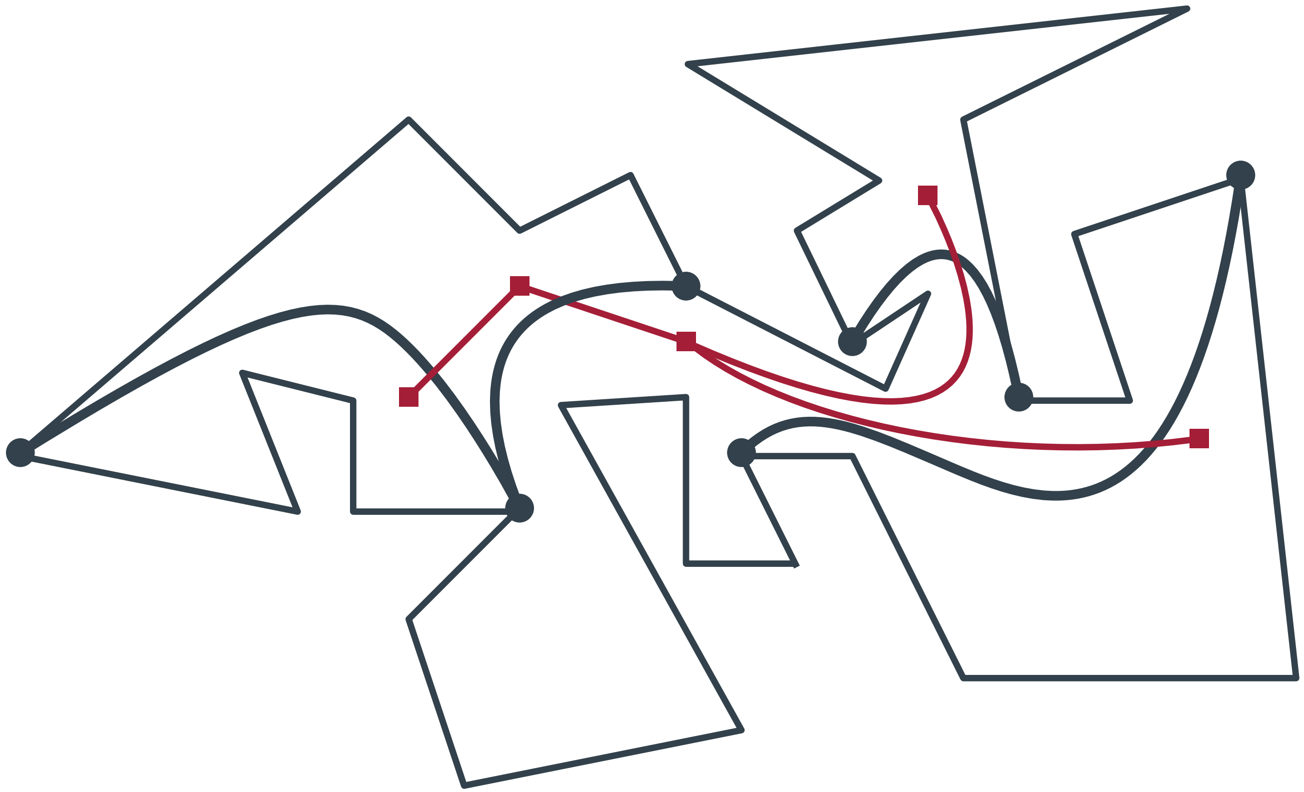
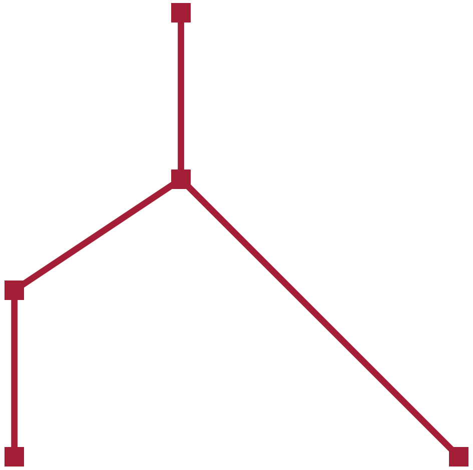


- Traverse the dual tree twice
- Bottom-up: refine the polygon
  - choose reflex edges over convex edges (corresponding to leaves)
- Top-down: place bend points in the refined polygons



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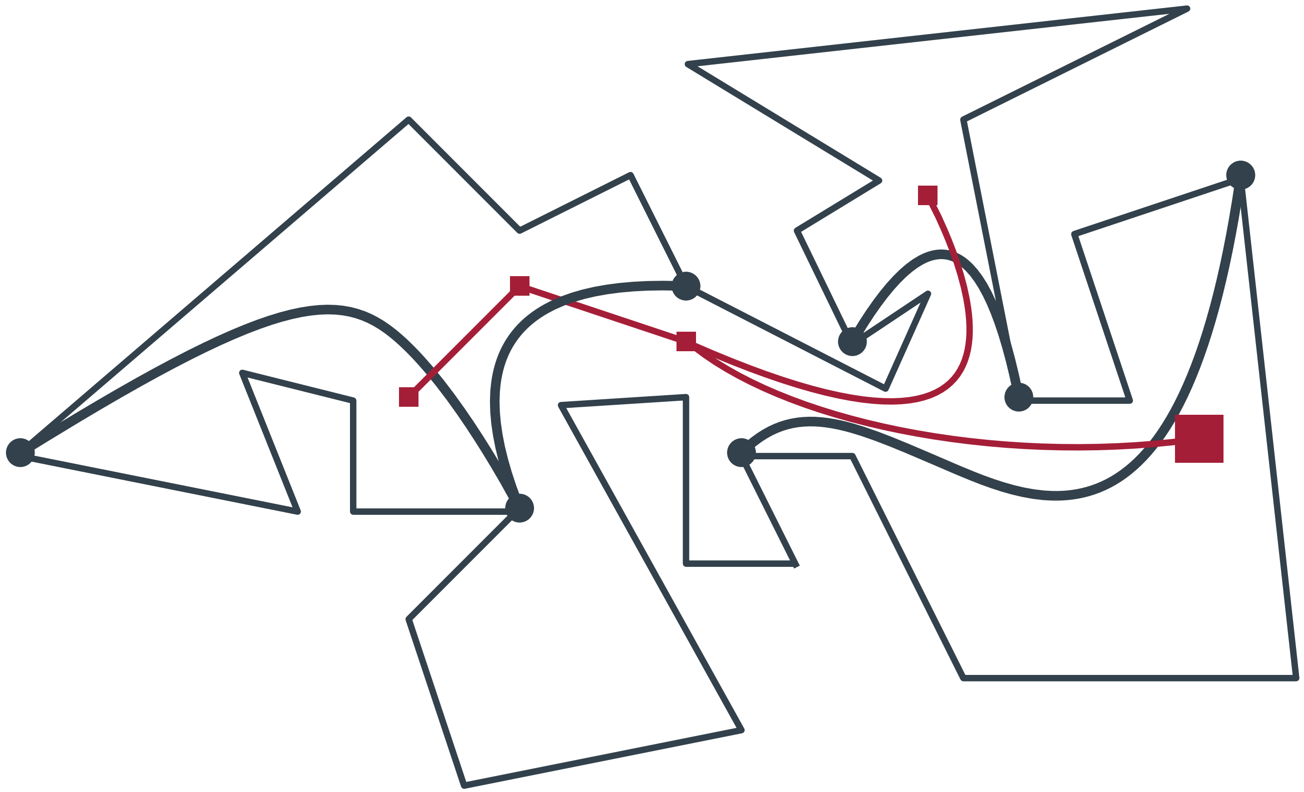

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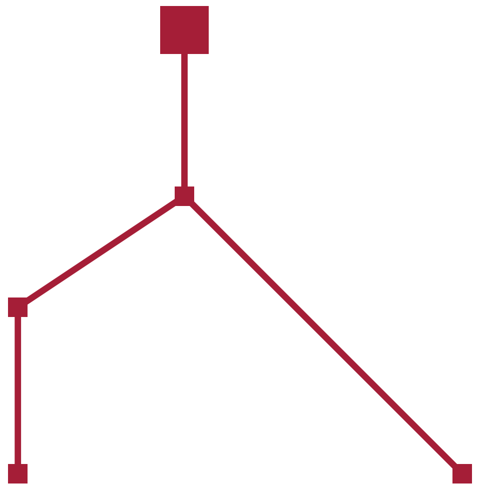

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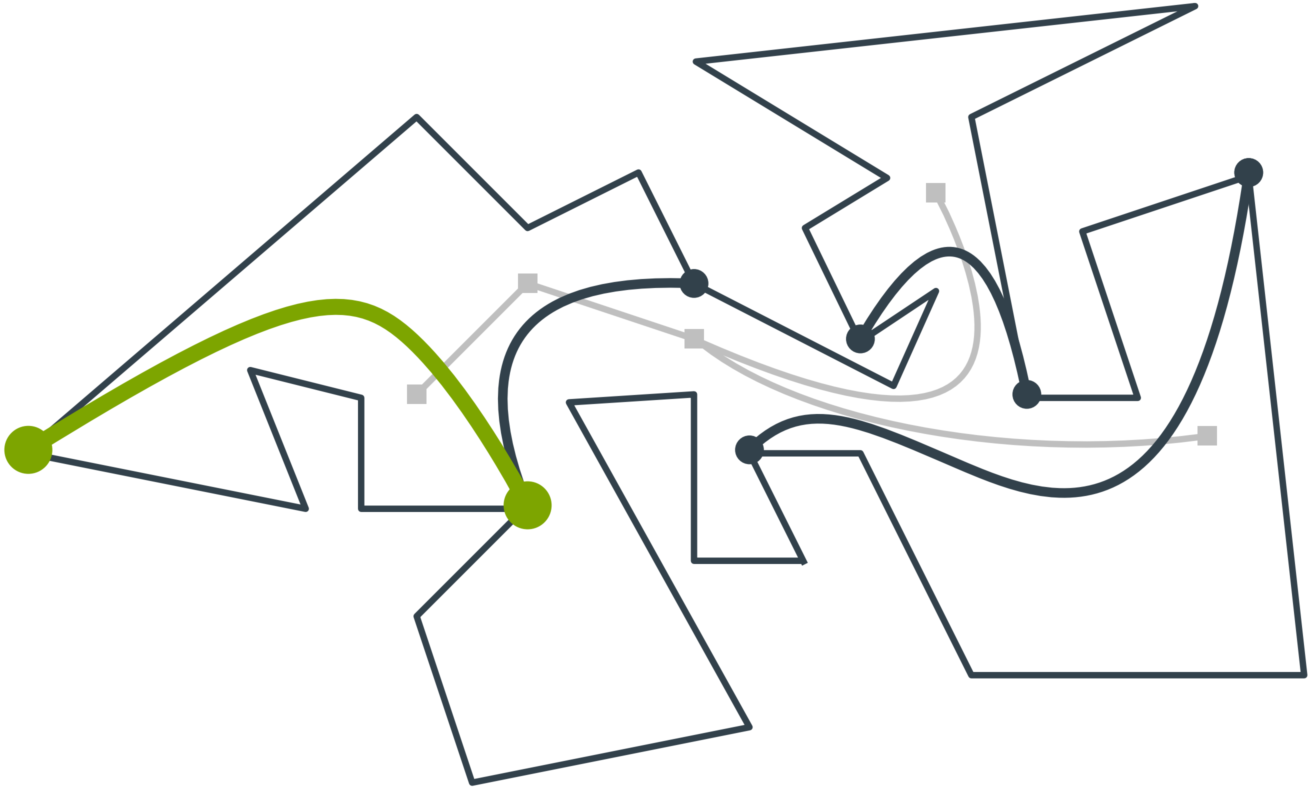

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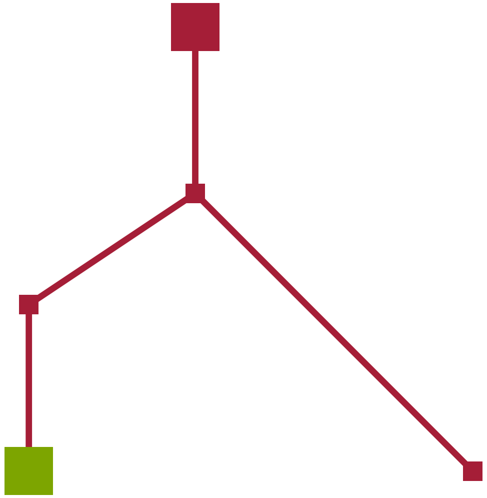

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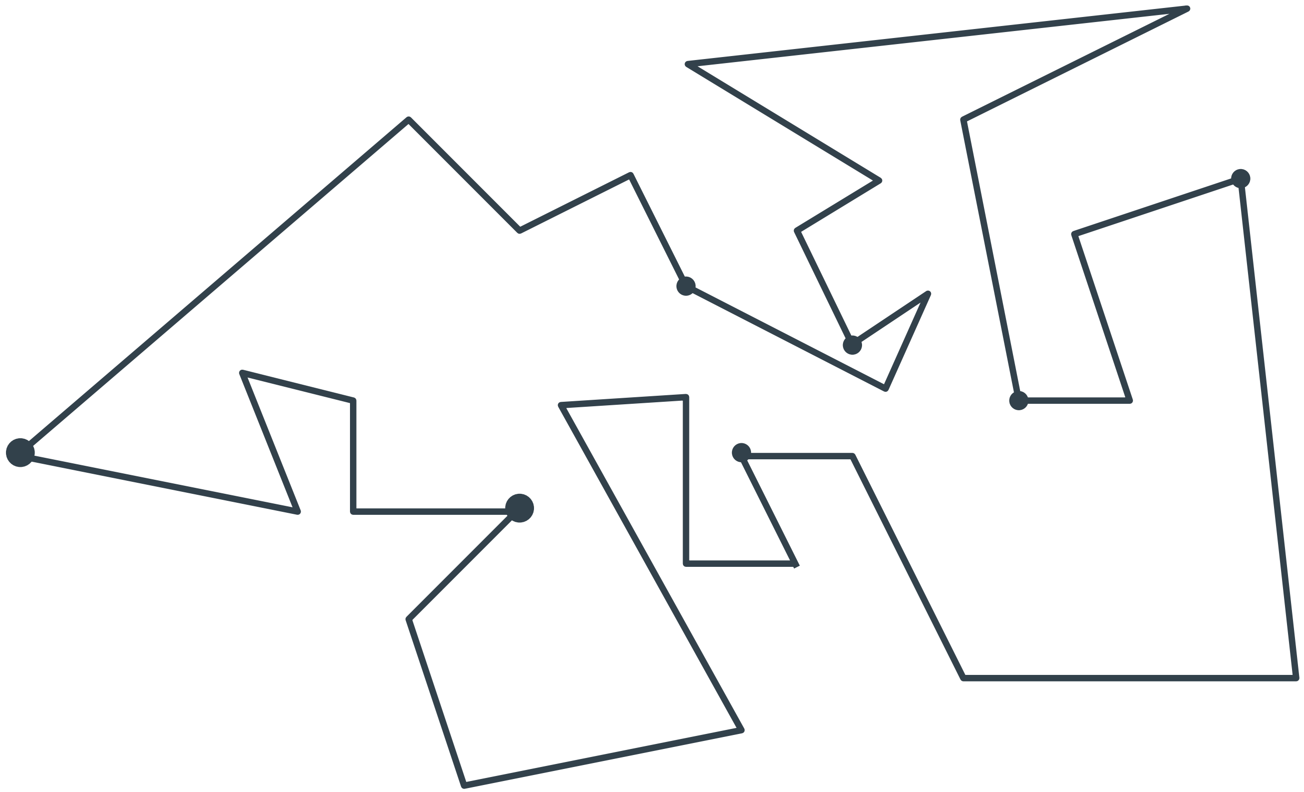

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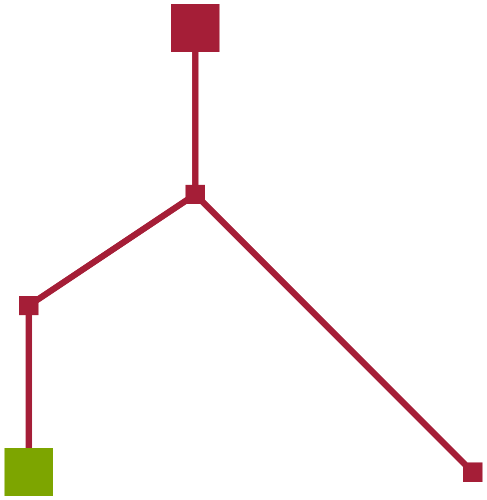

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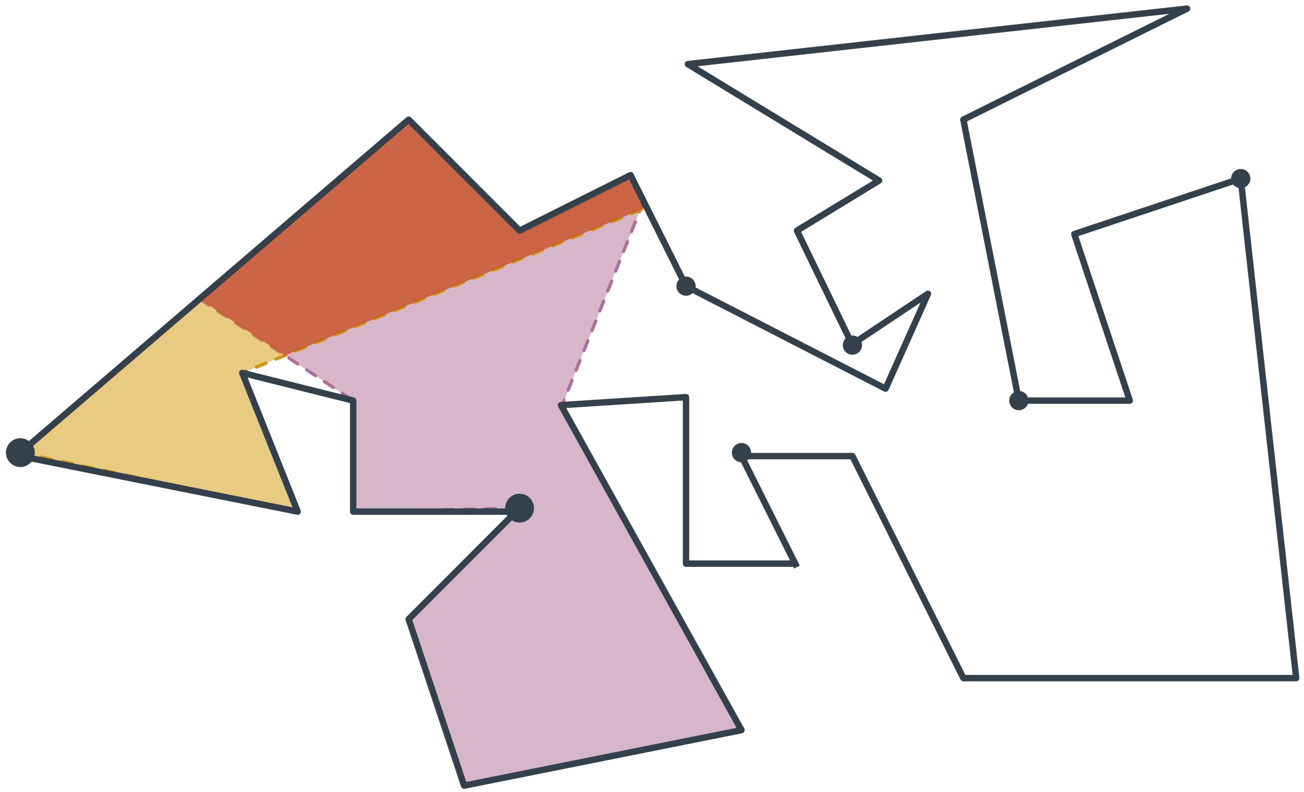

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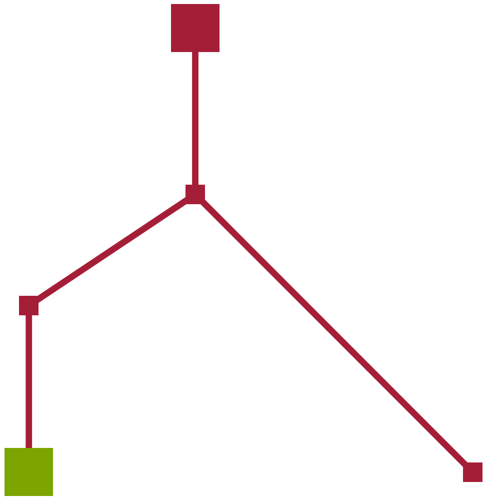

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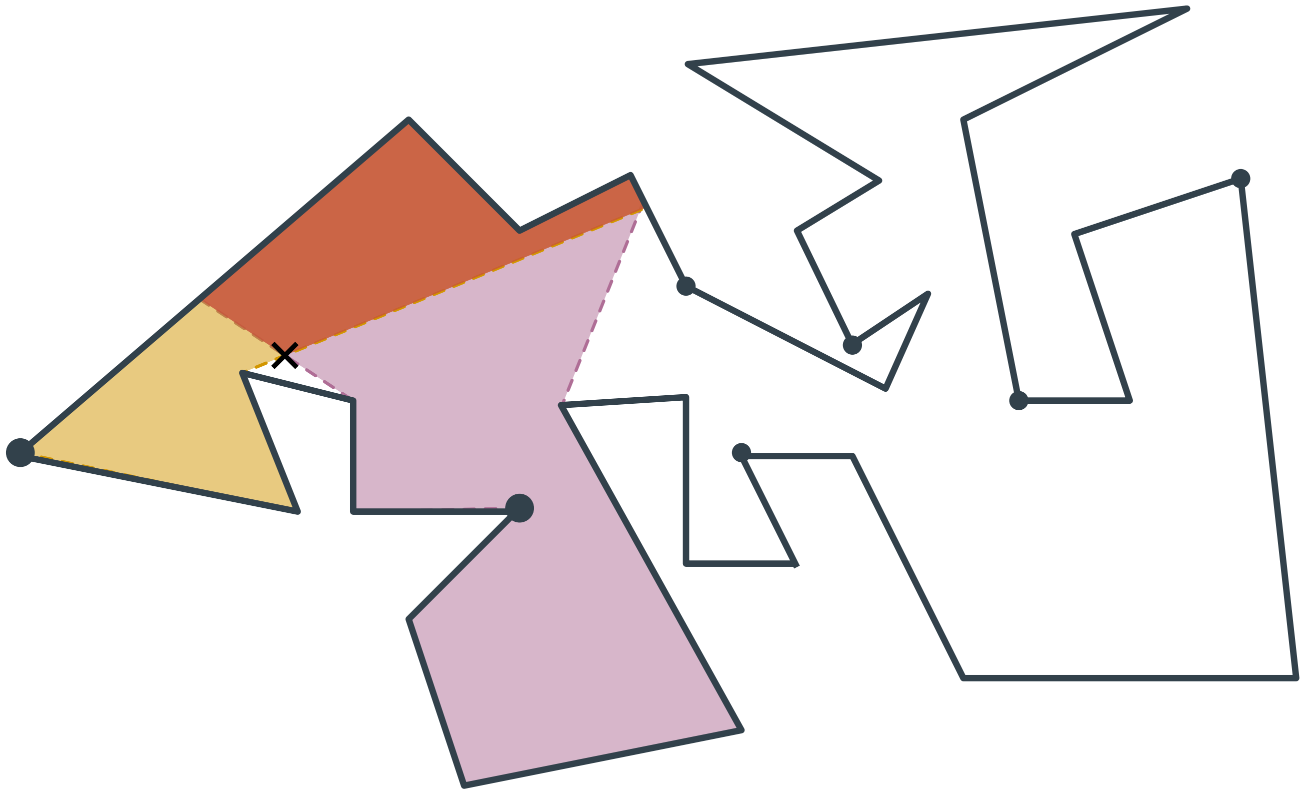

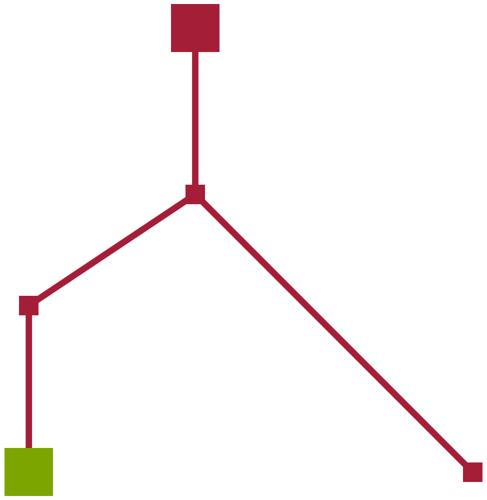

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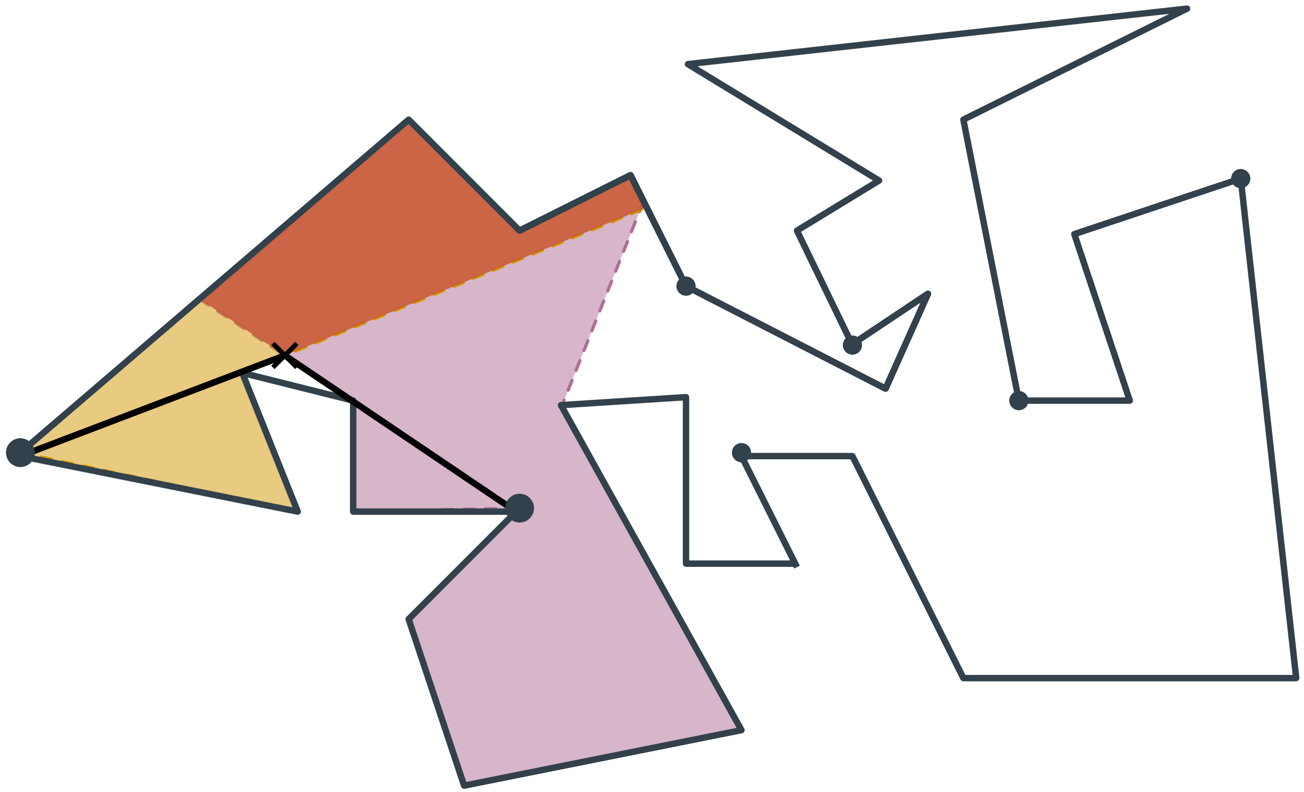

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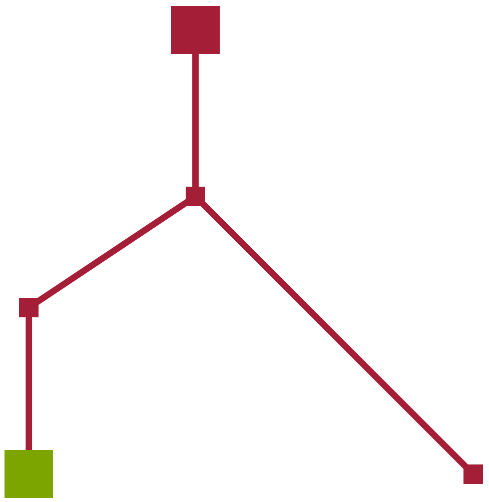

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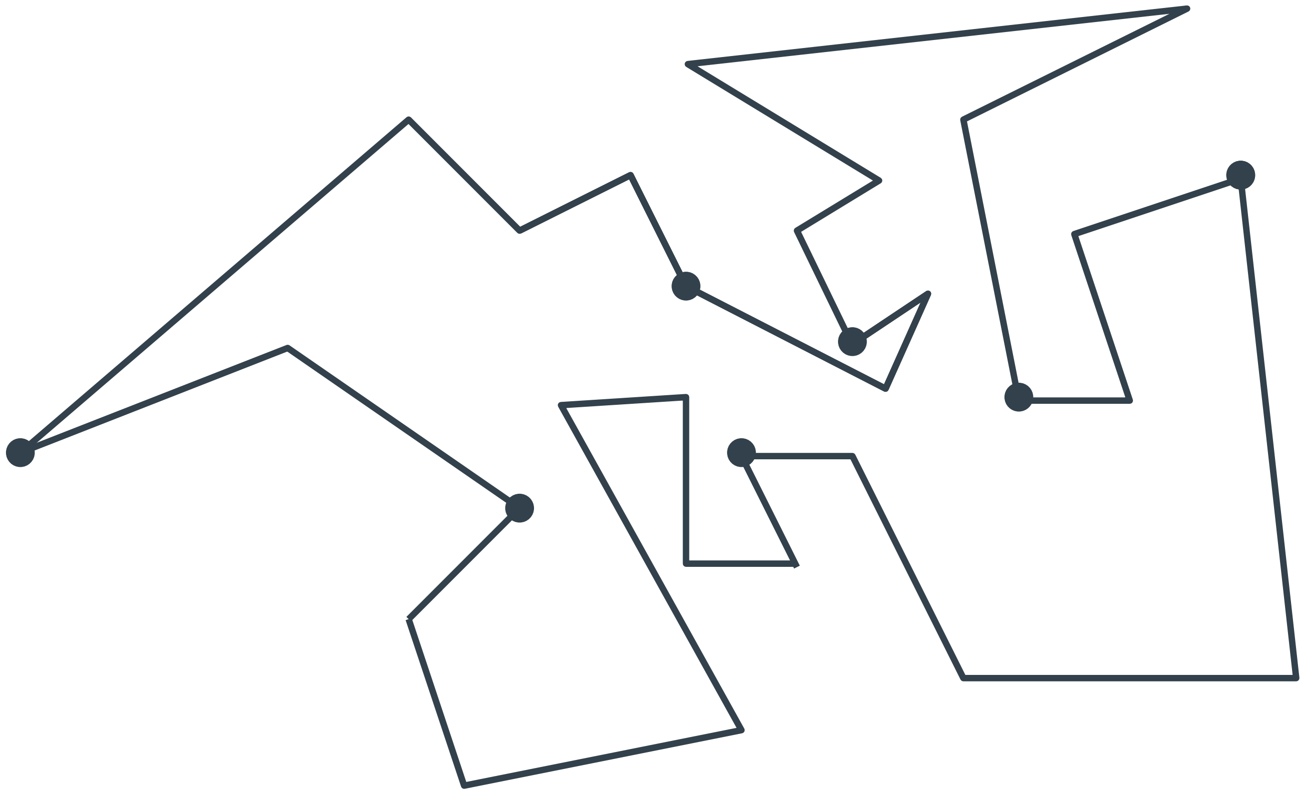

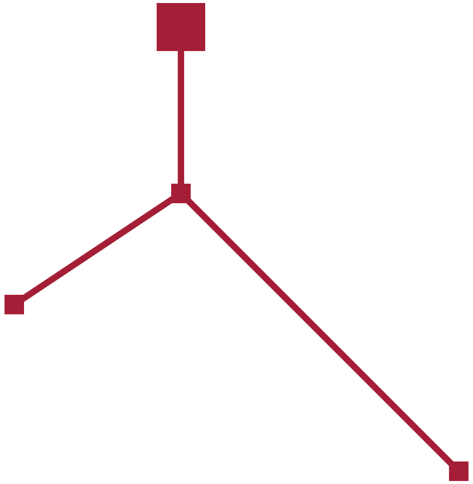

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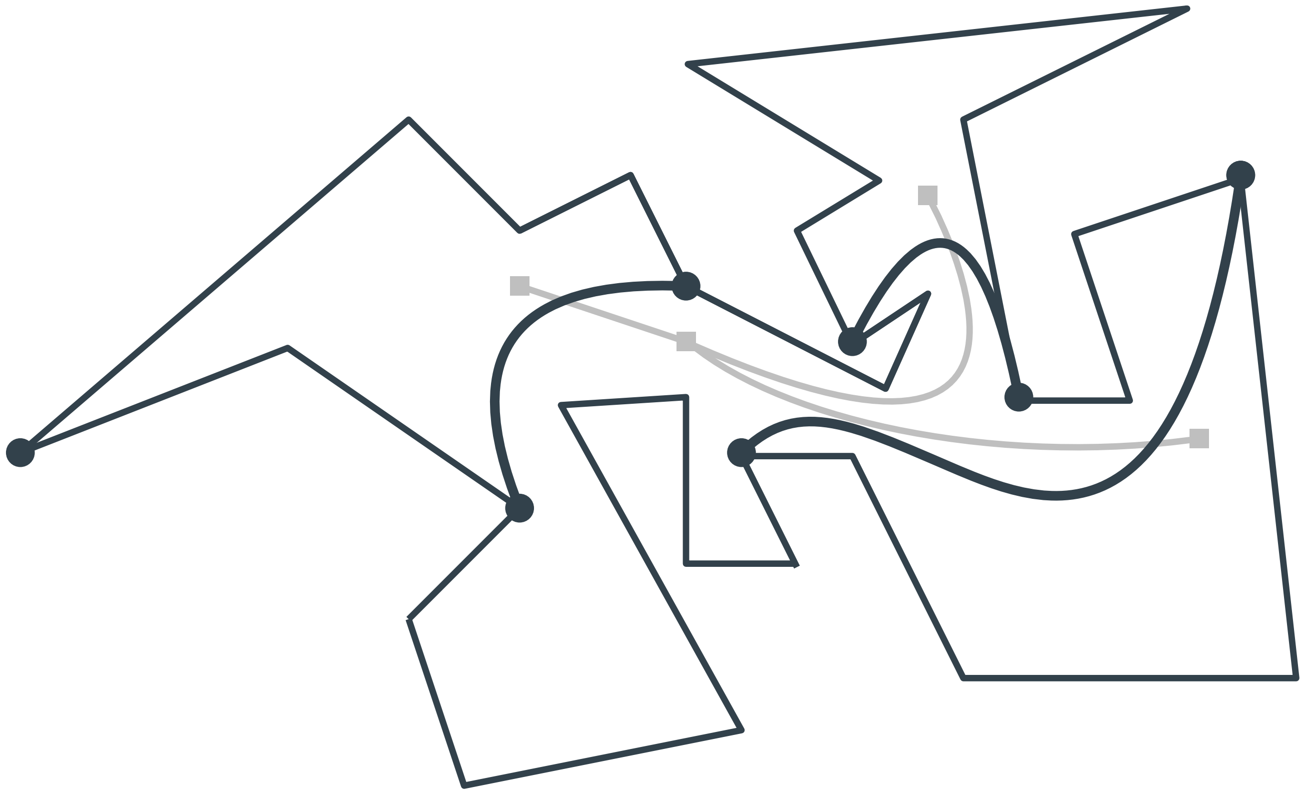

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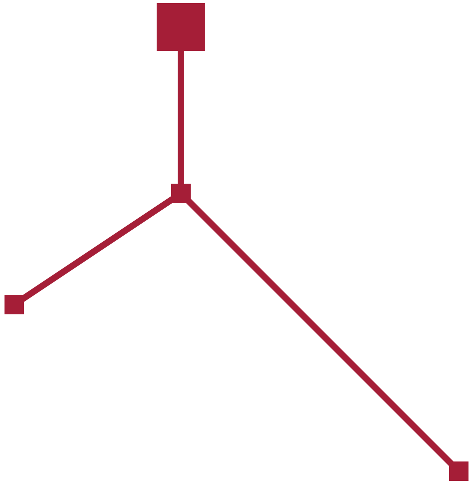

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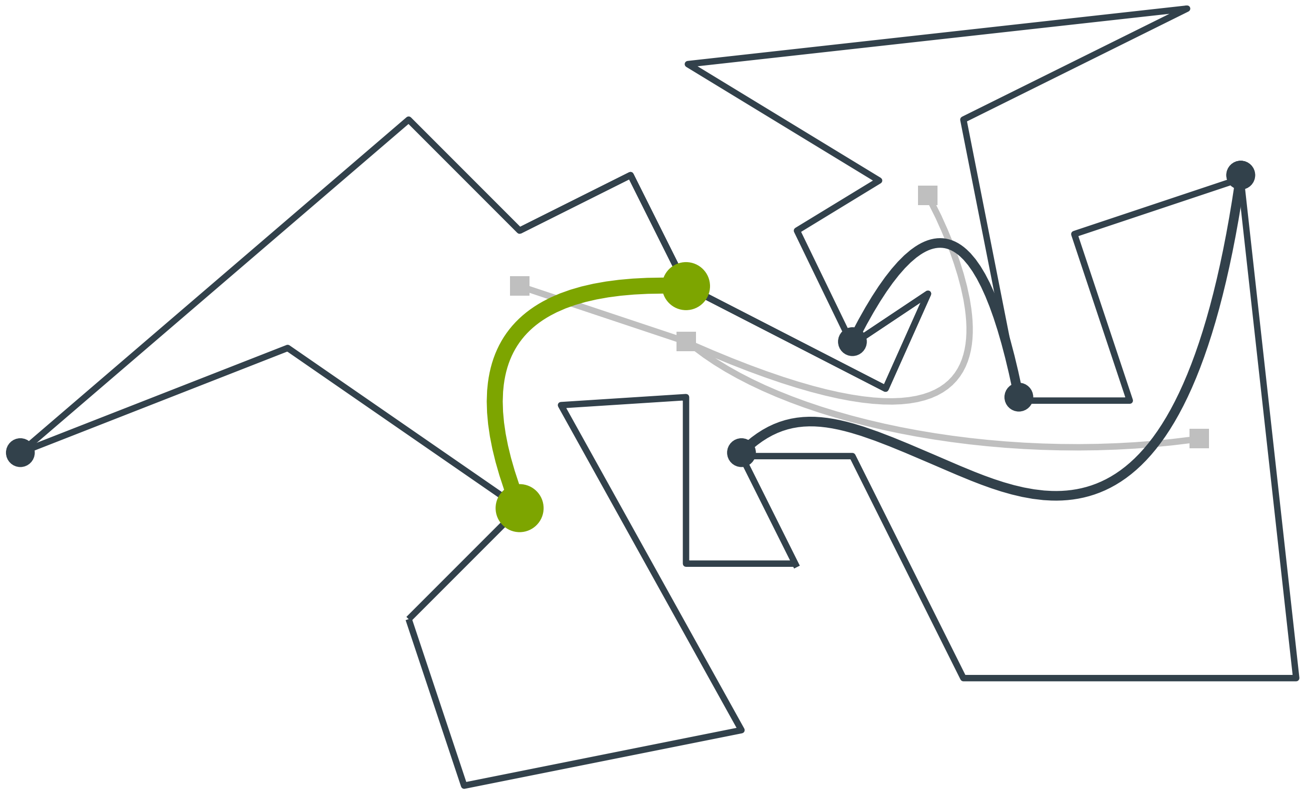

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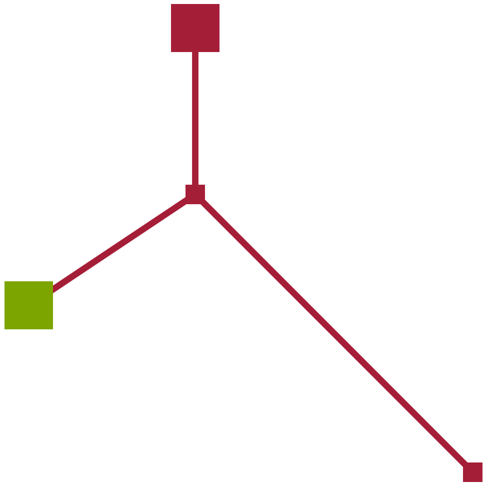

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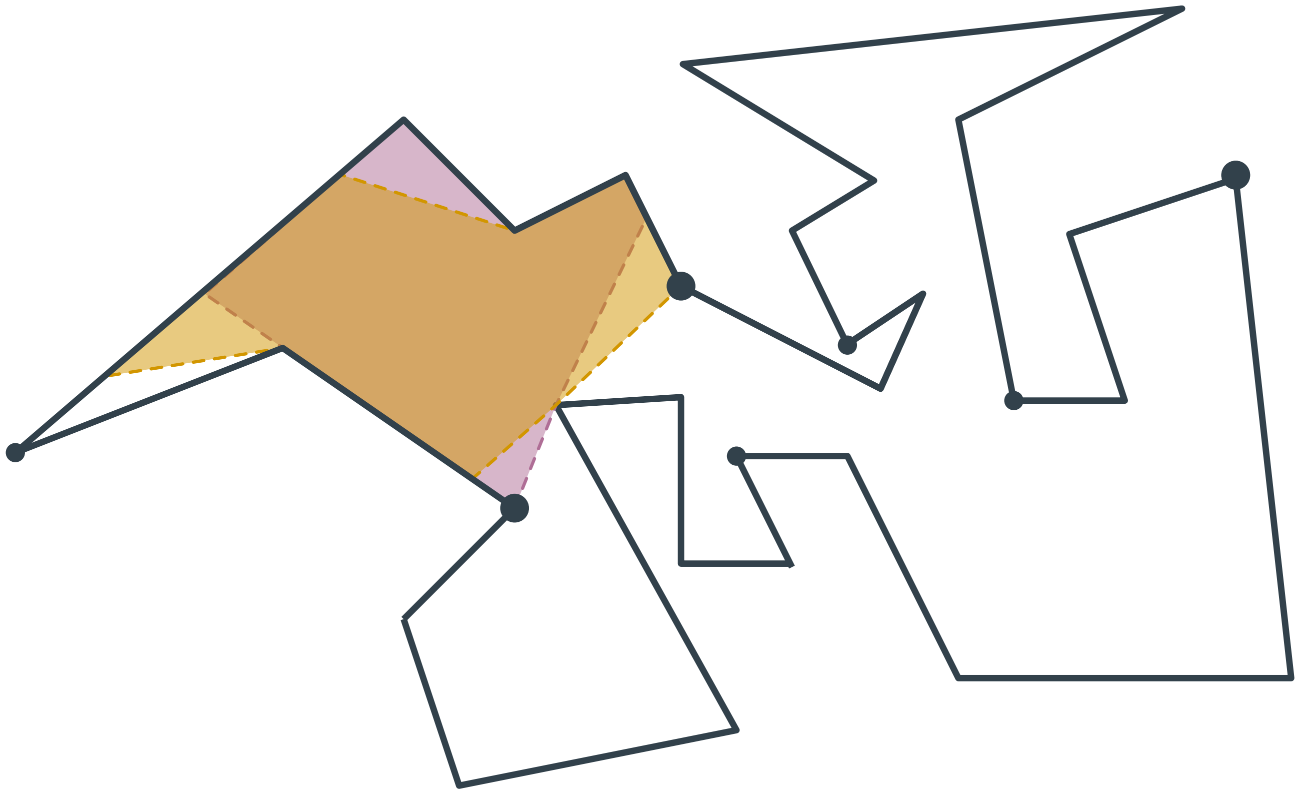

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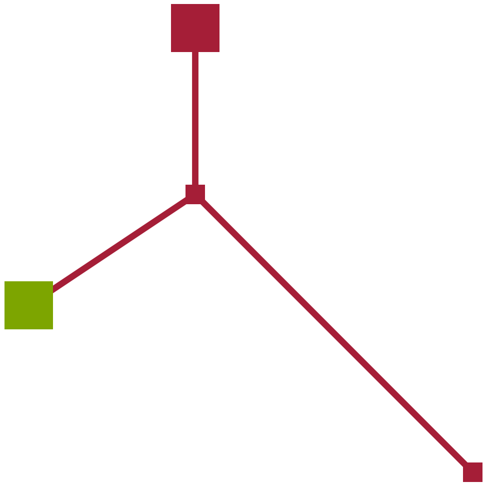

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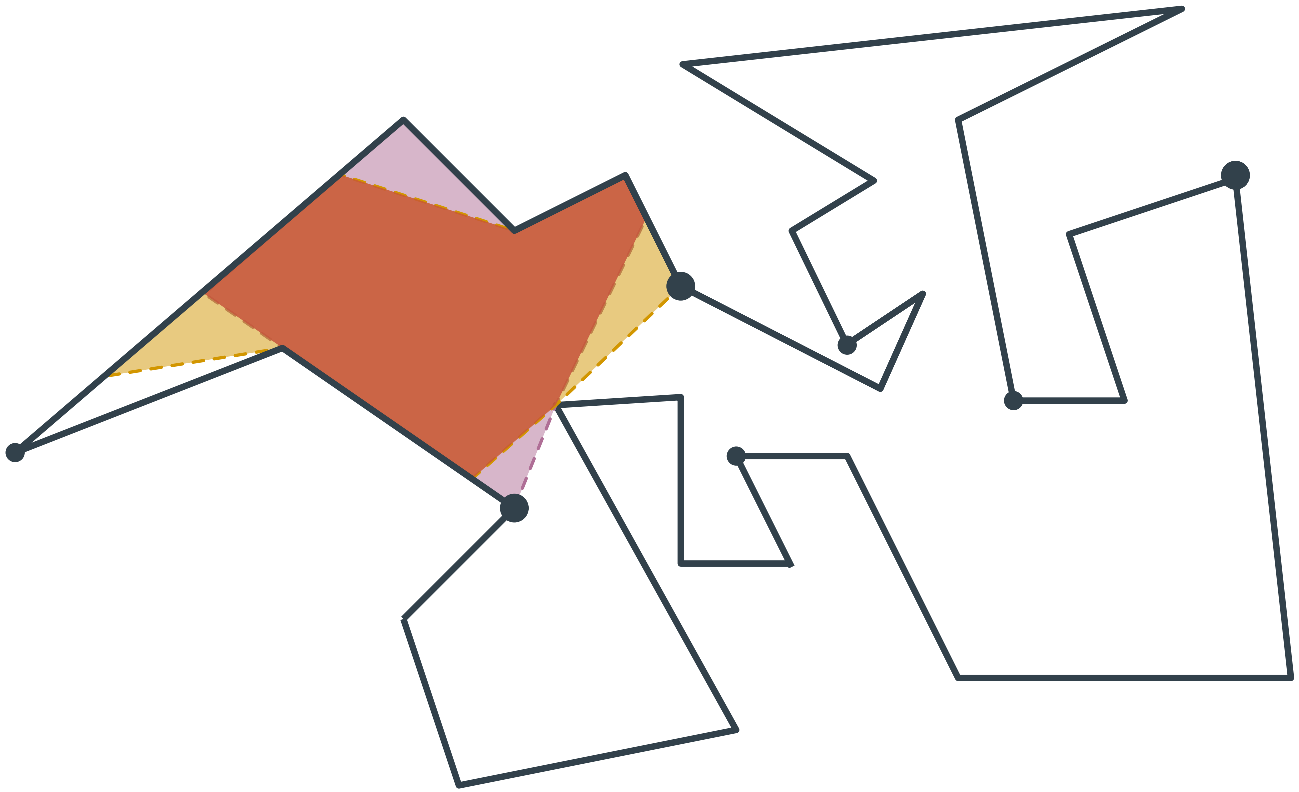

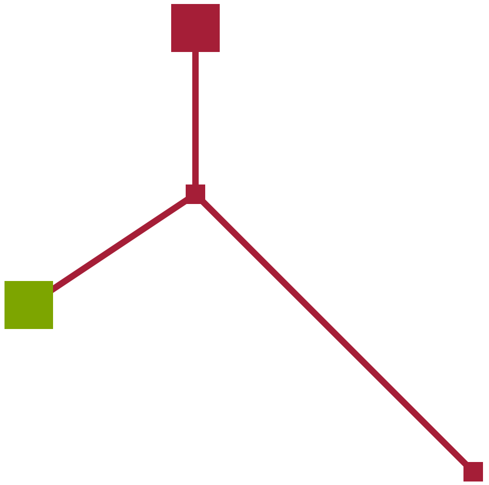

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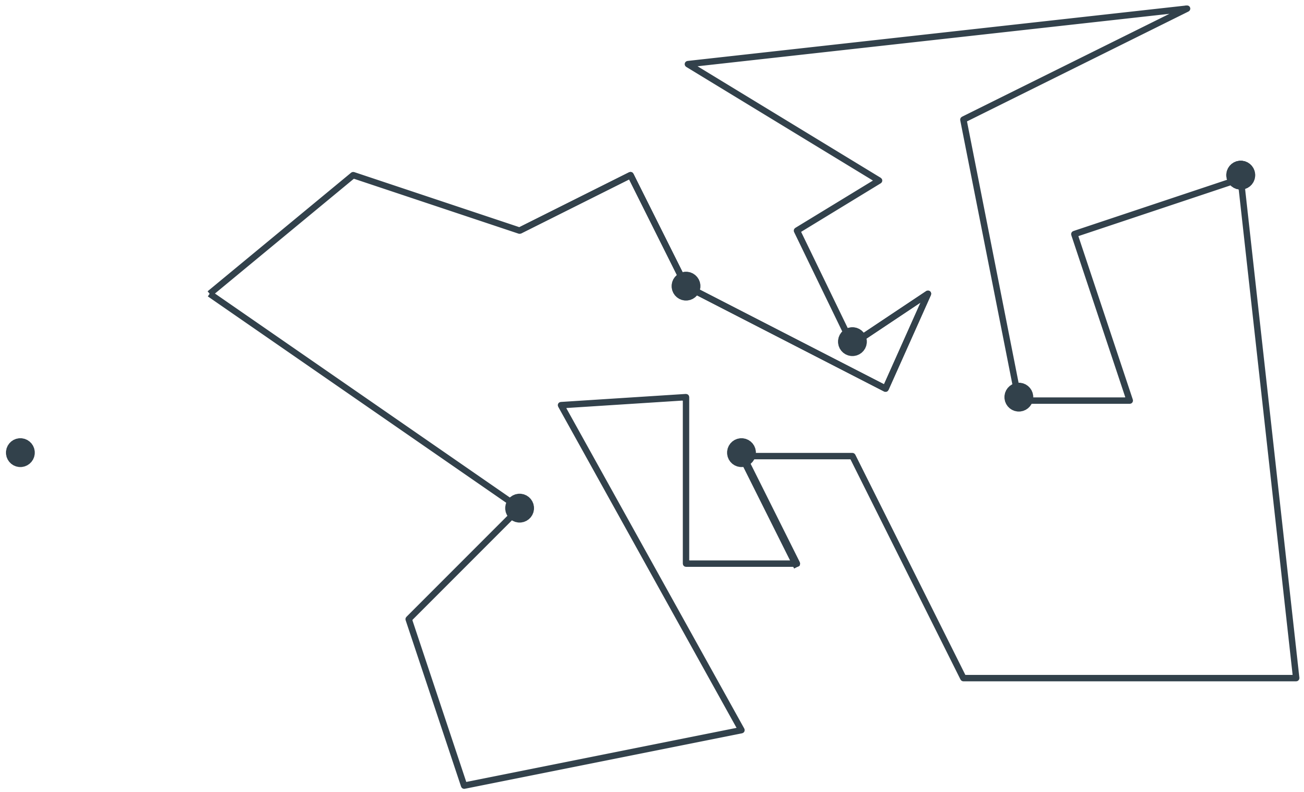

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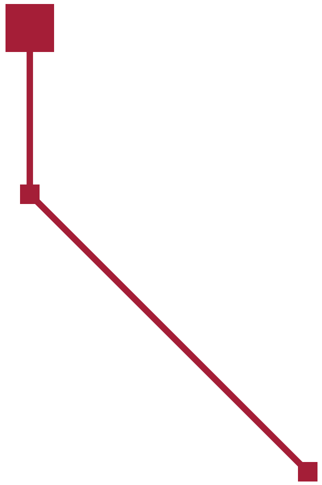

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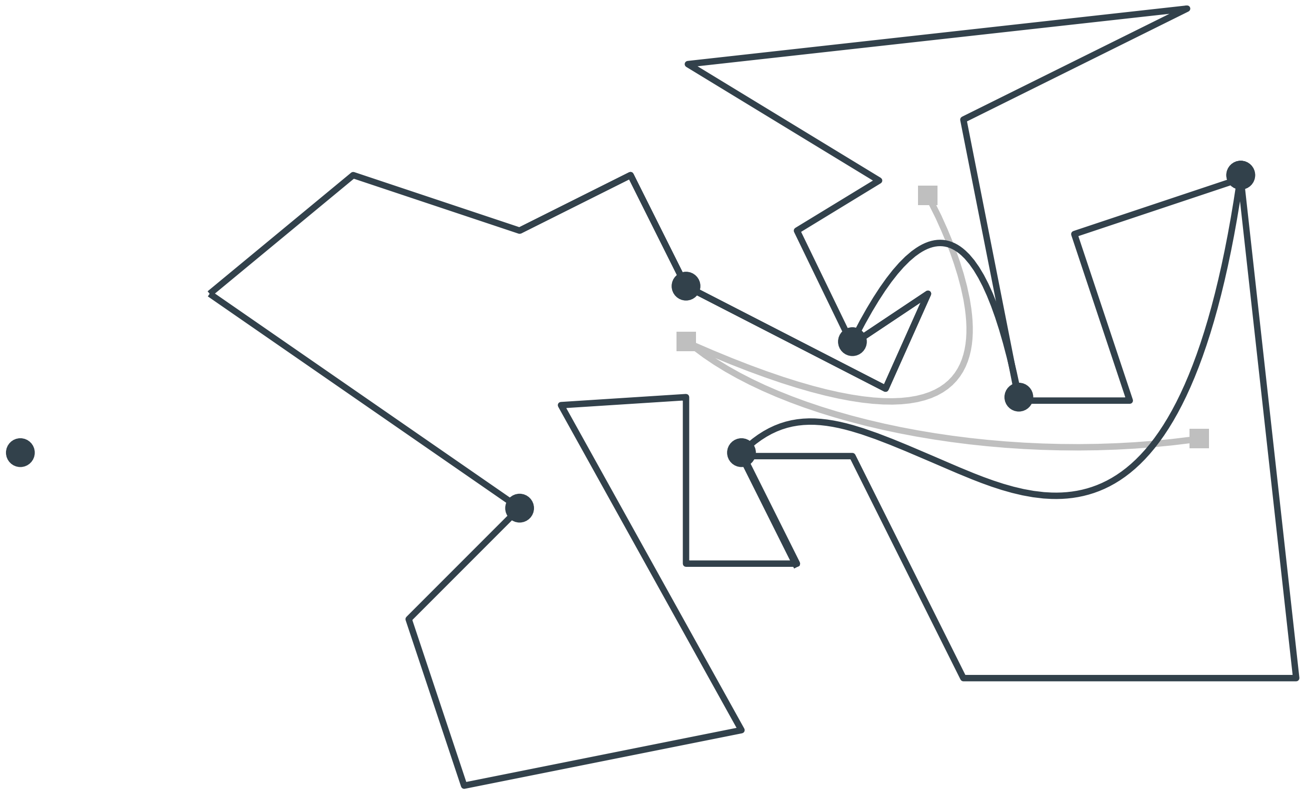

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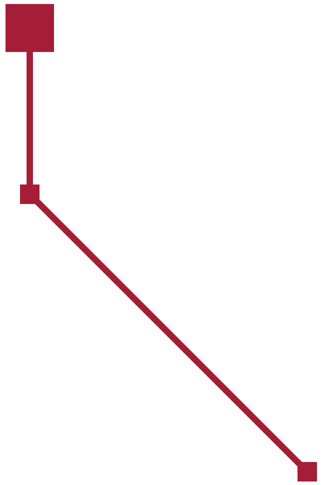

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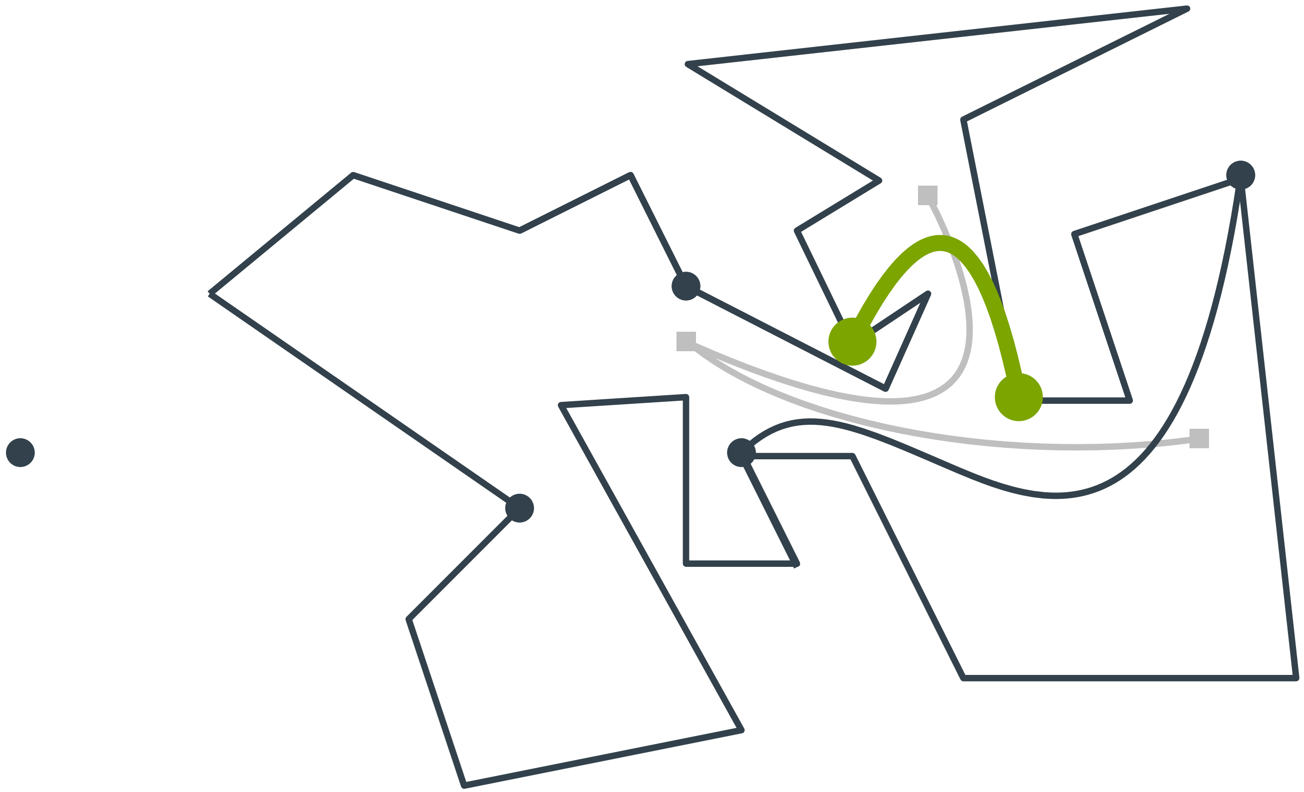

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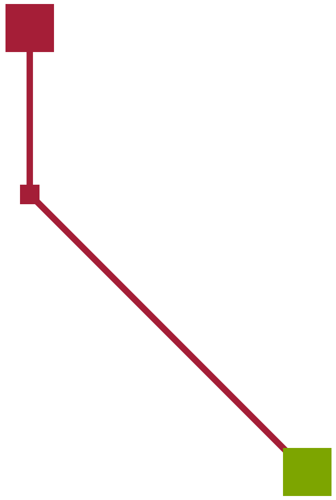

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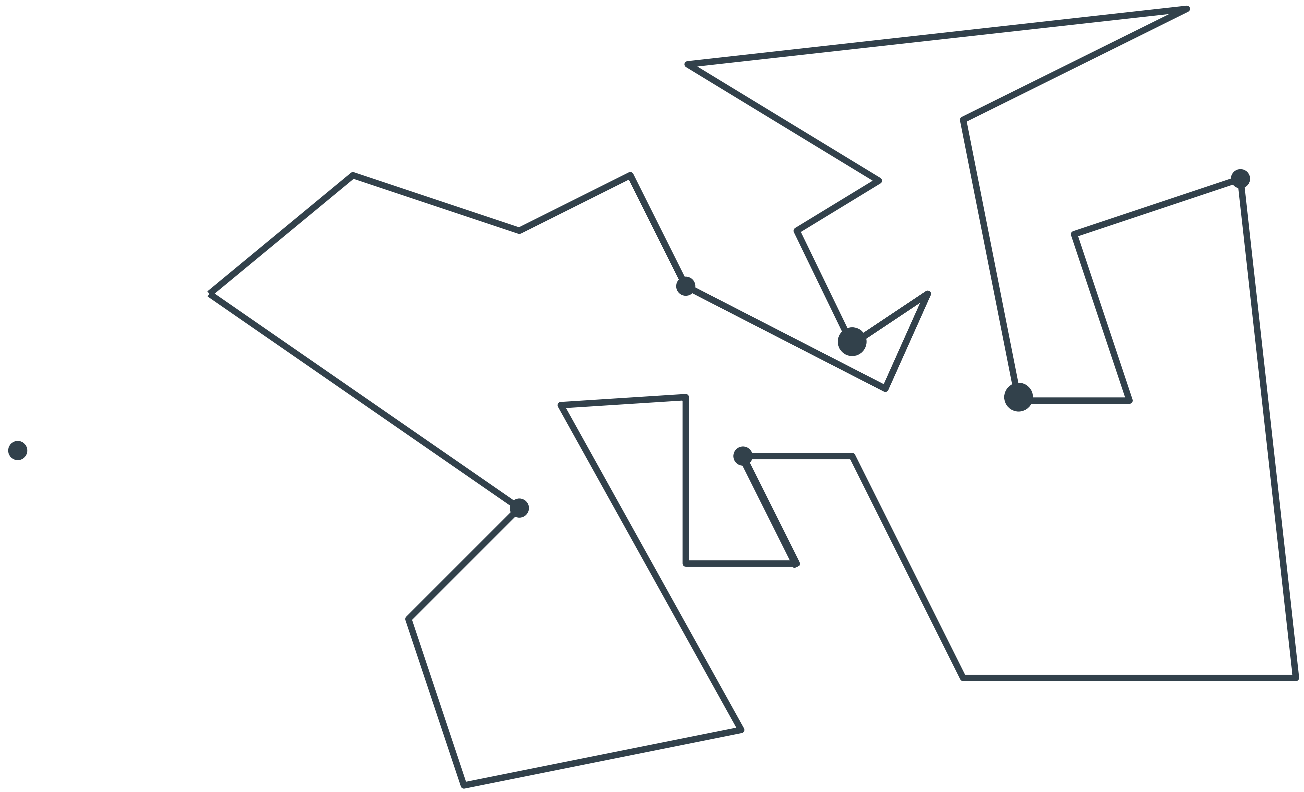

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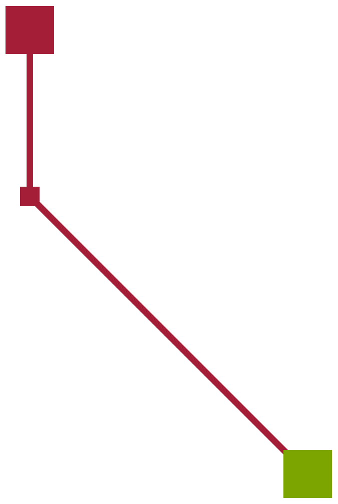

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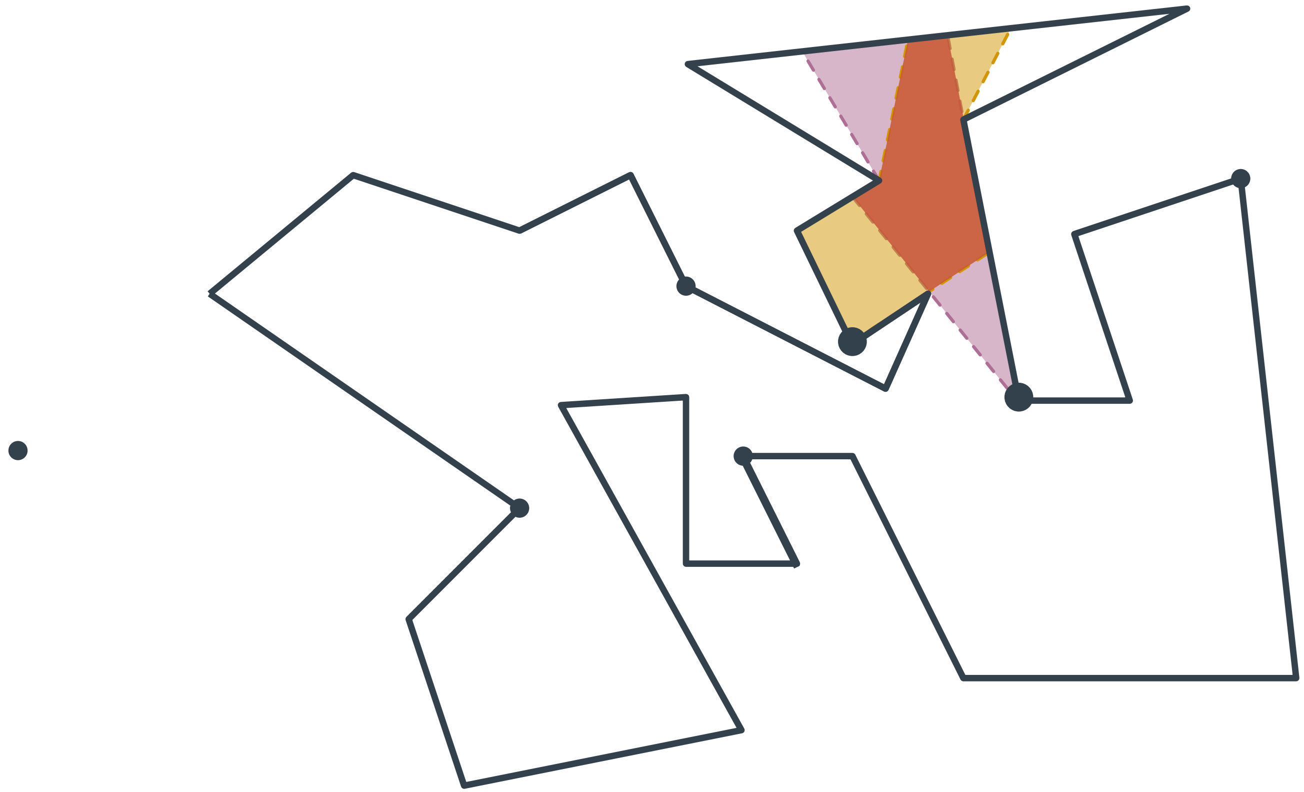

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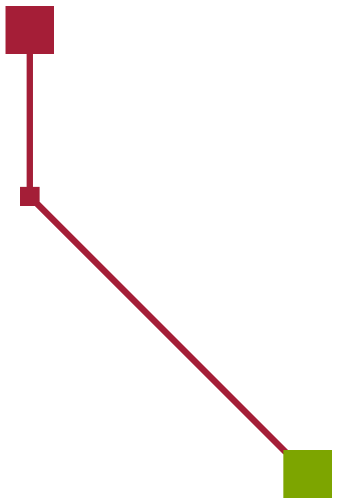

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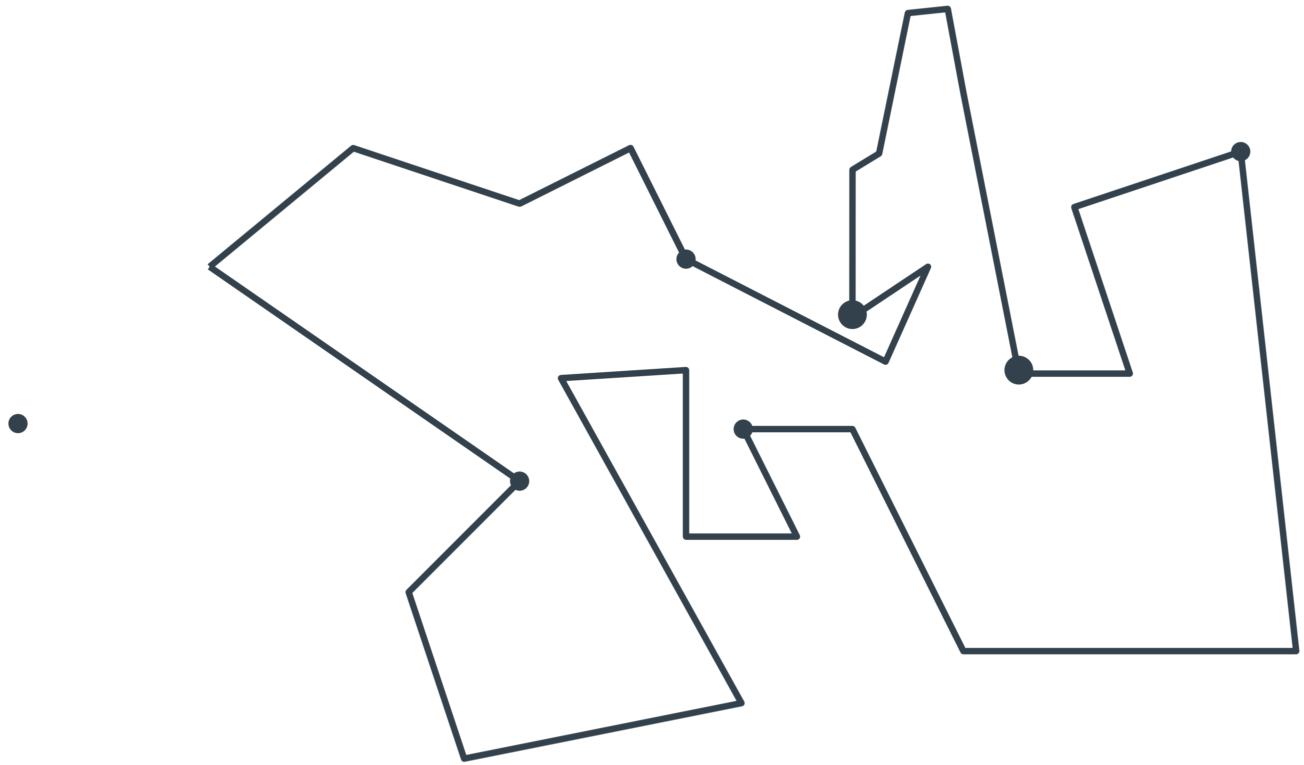

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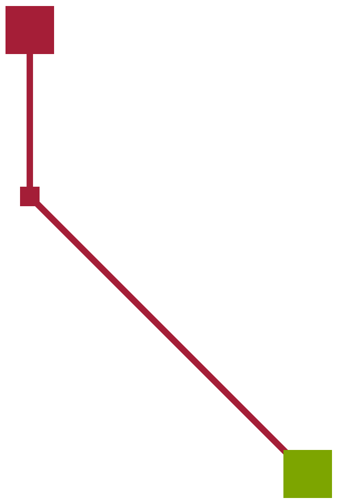

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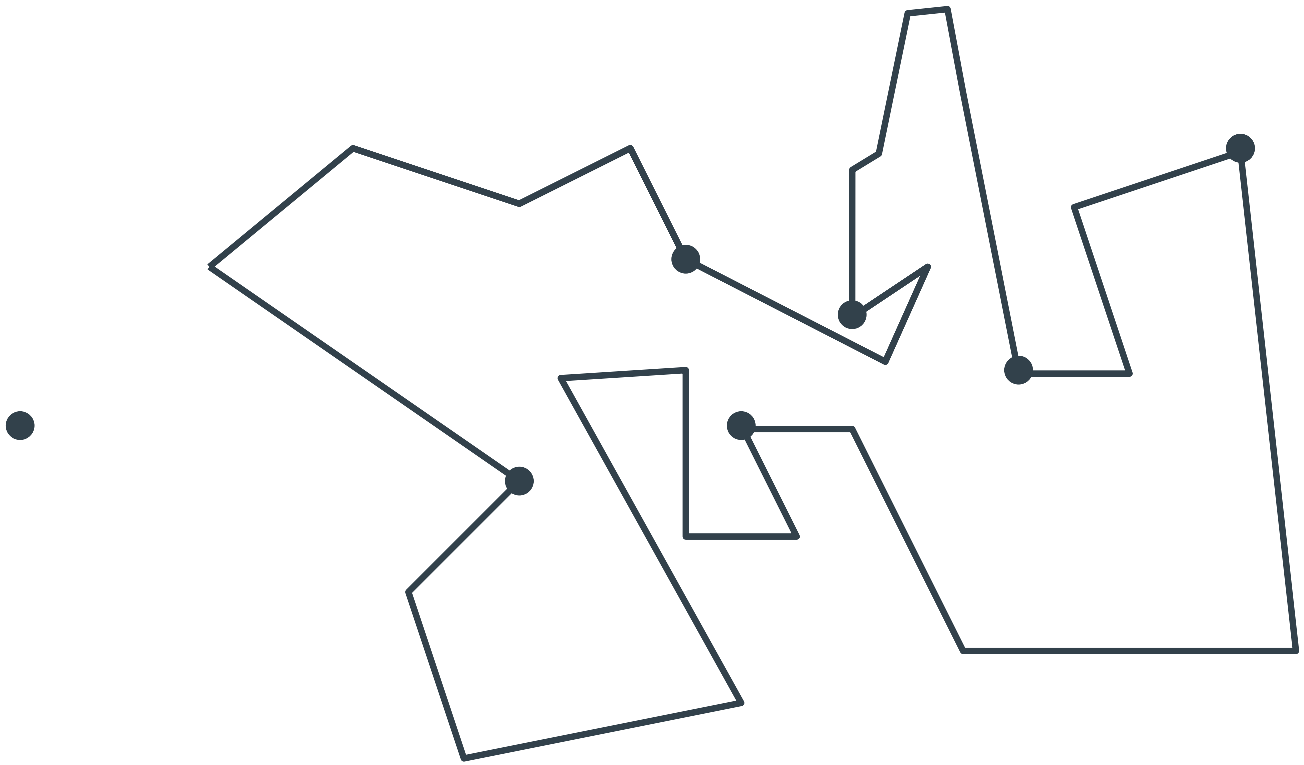

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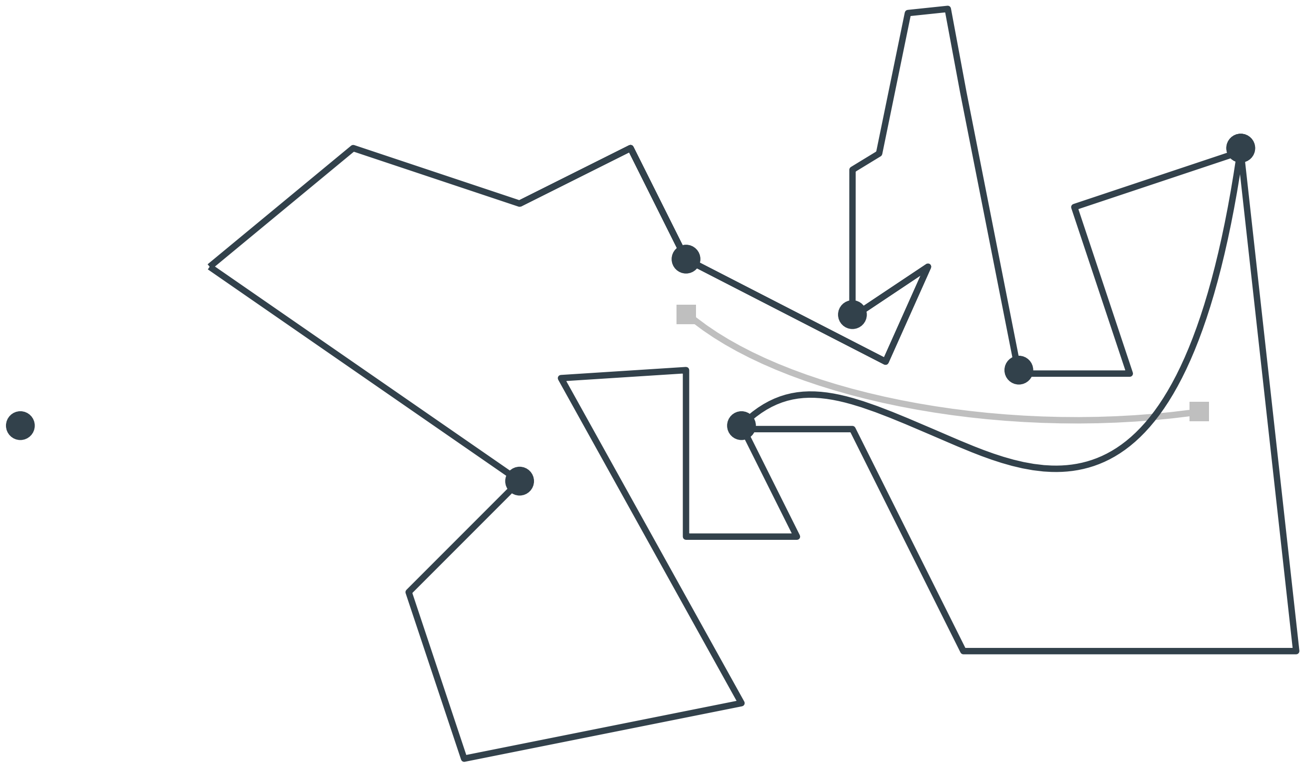

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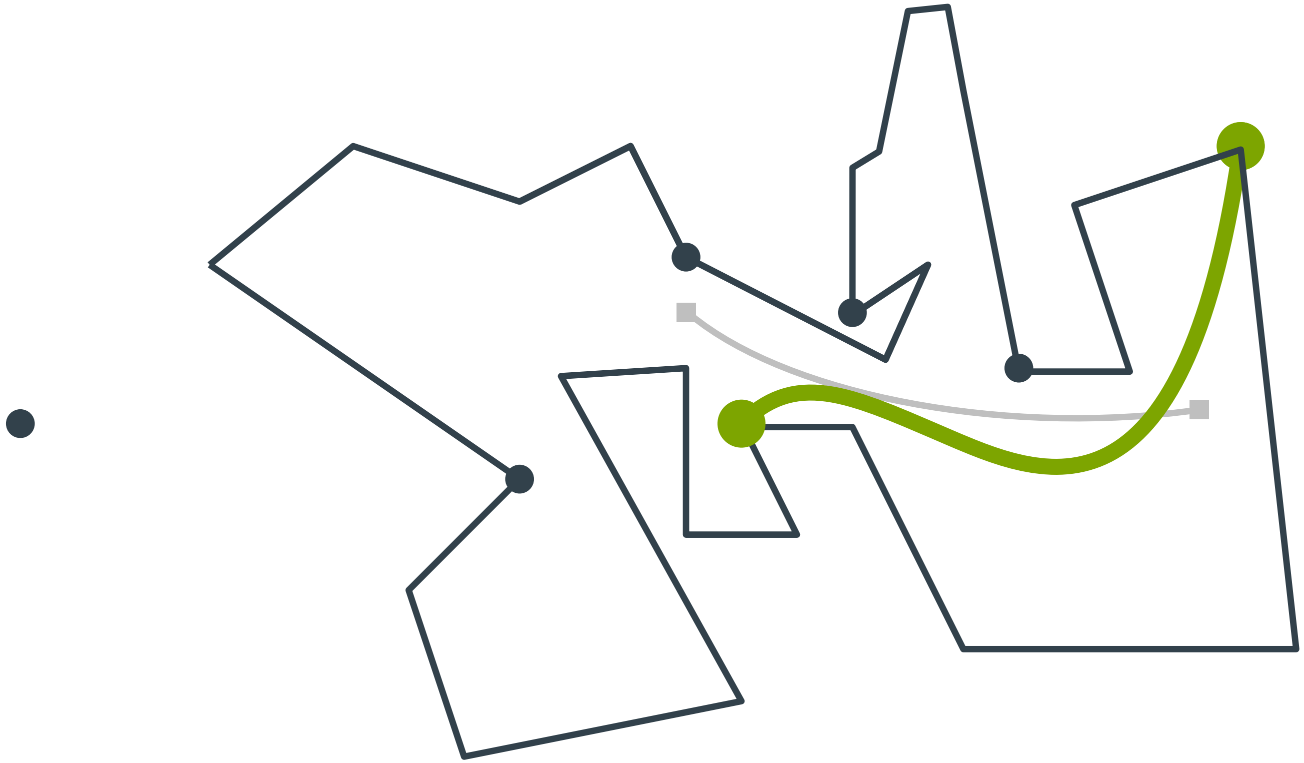

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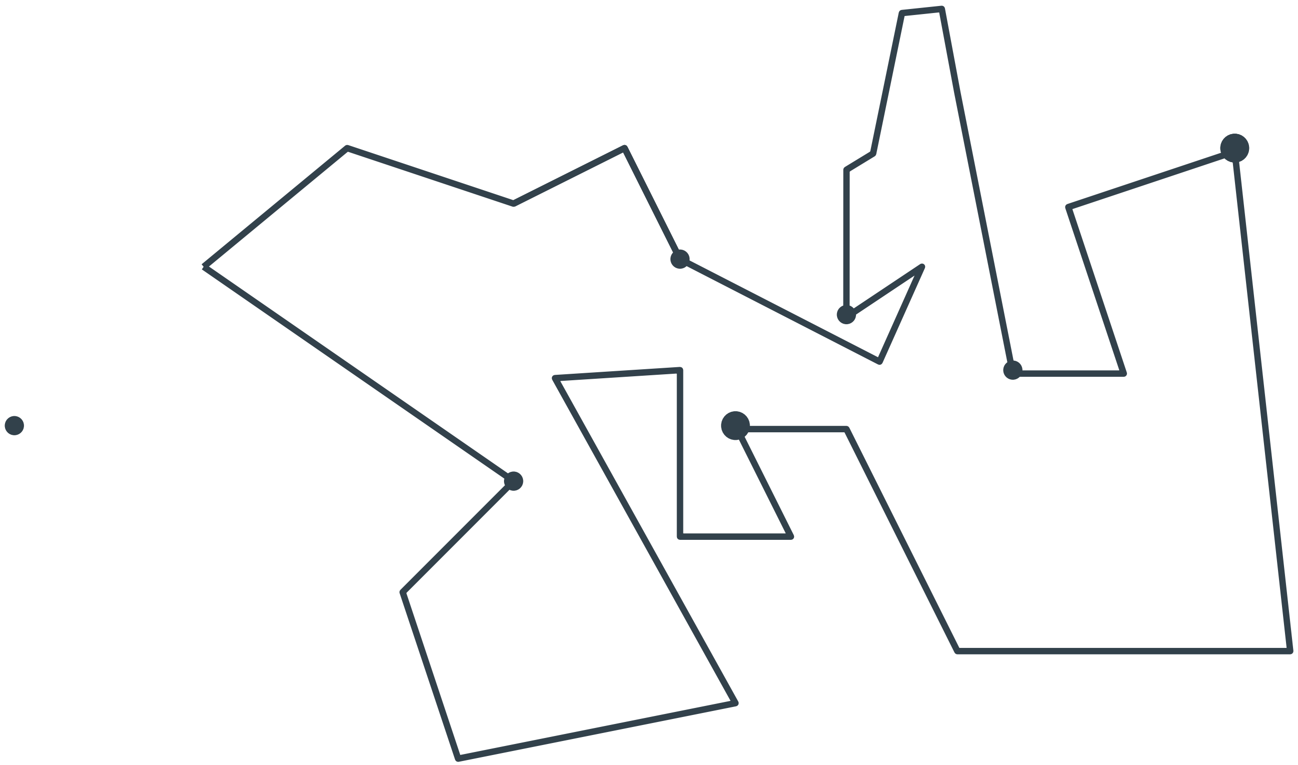

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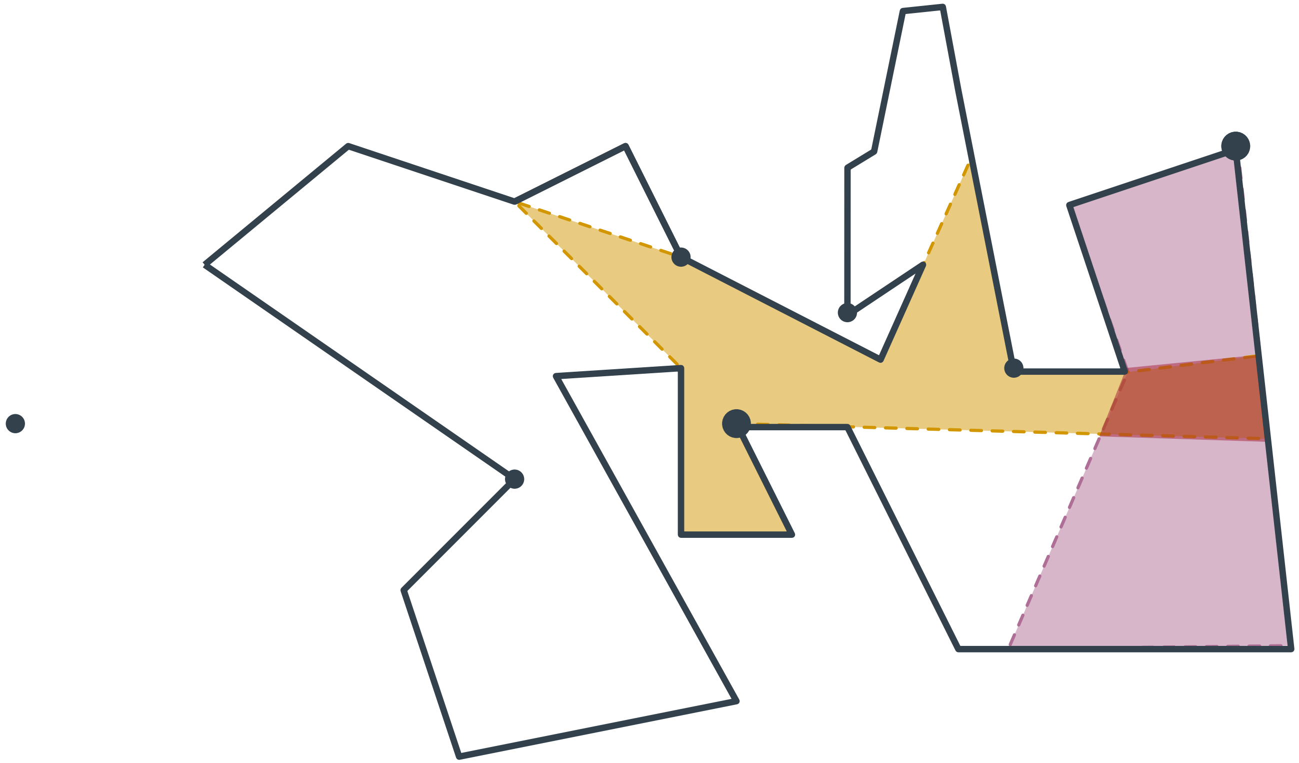

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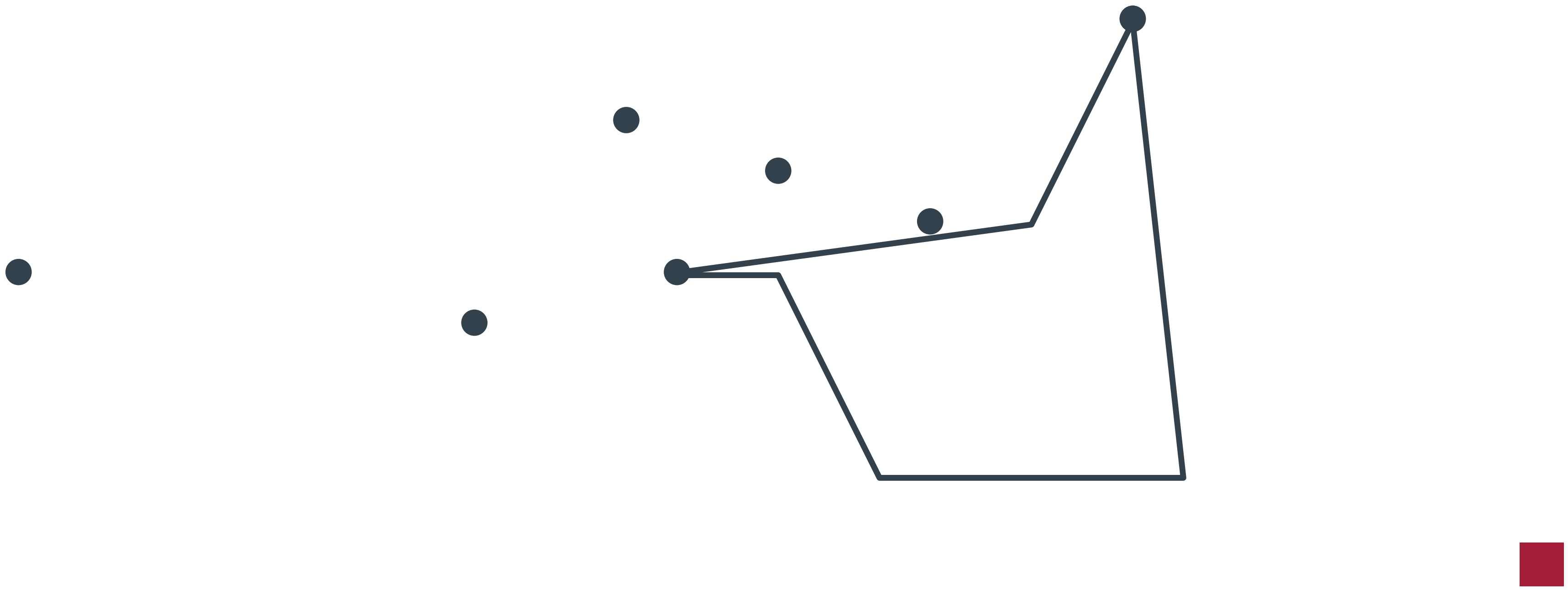

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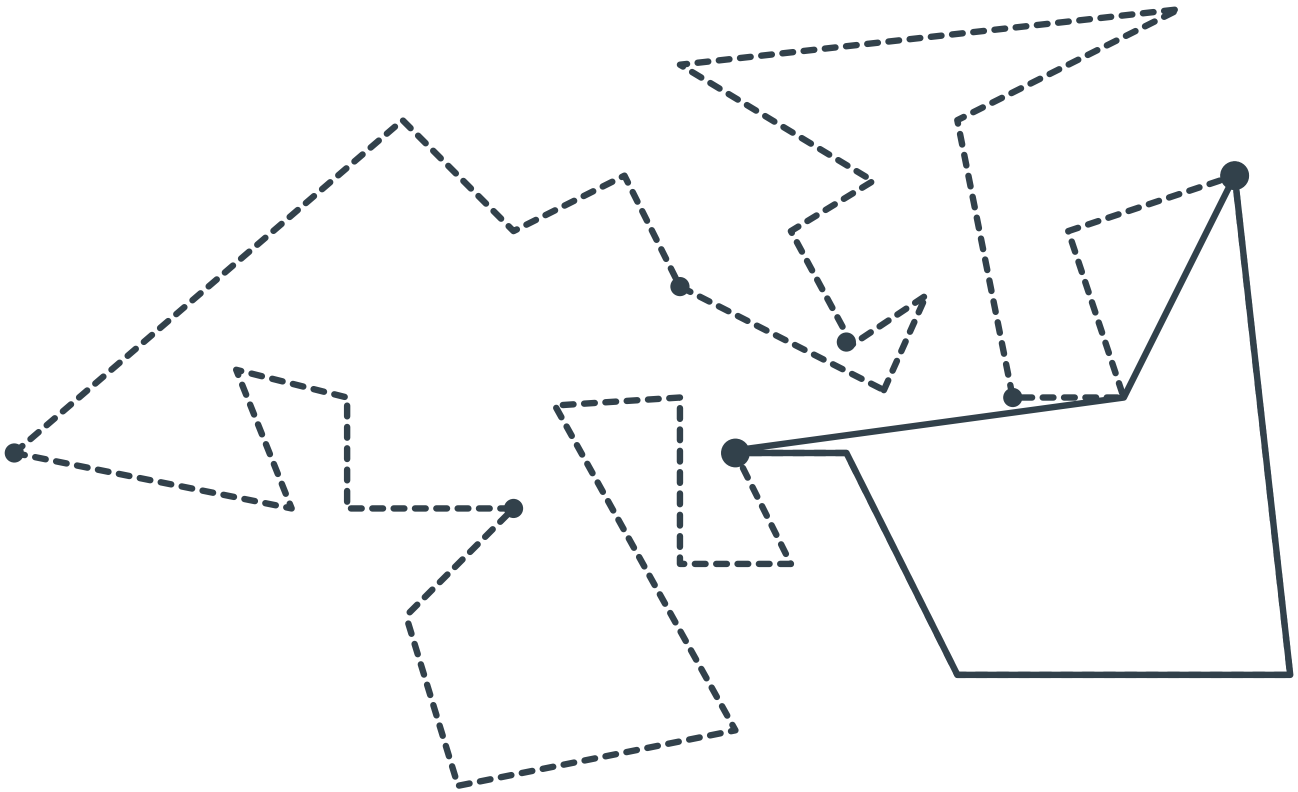

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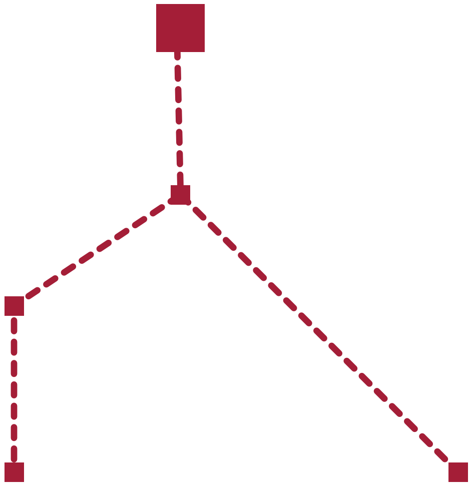

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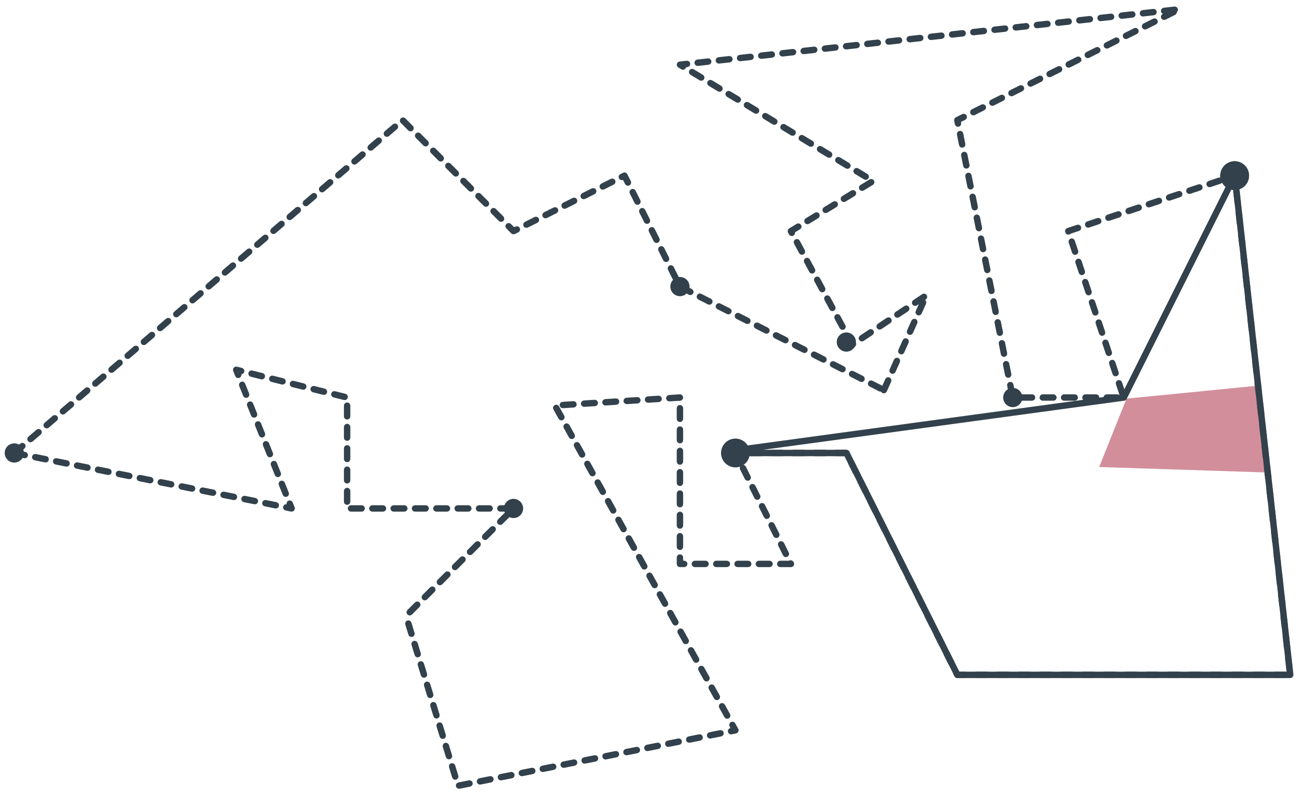

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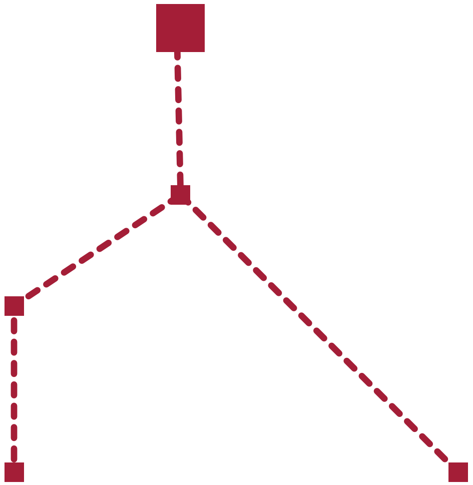

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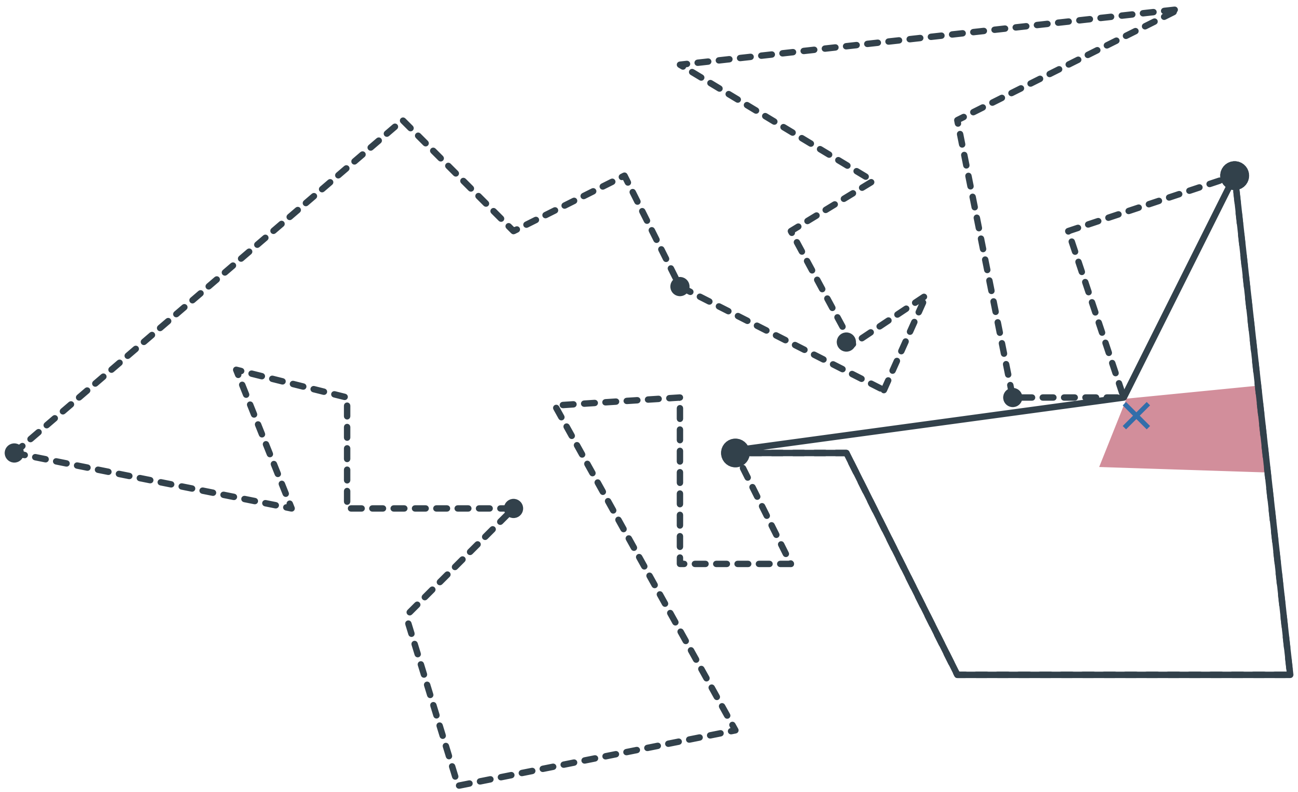

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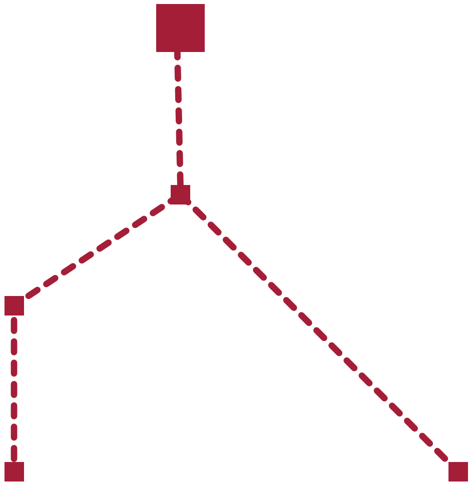

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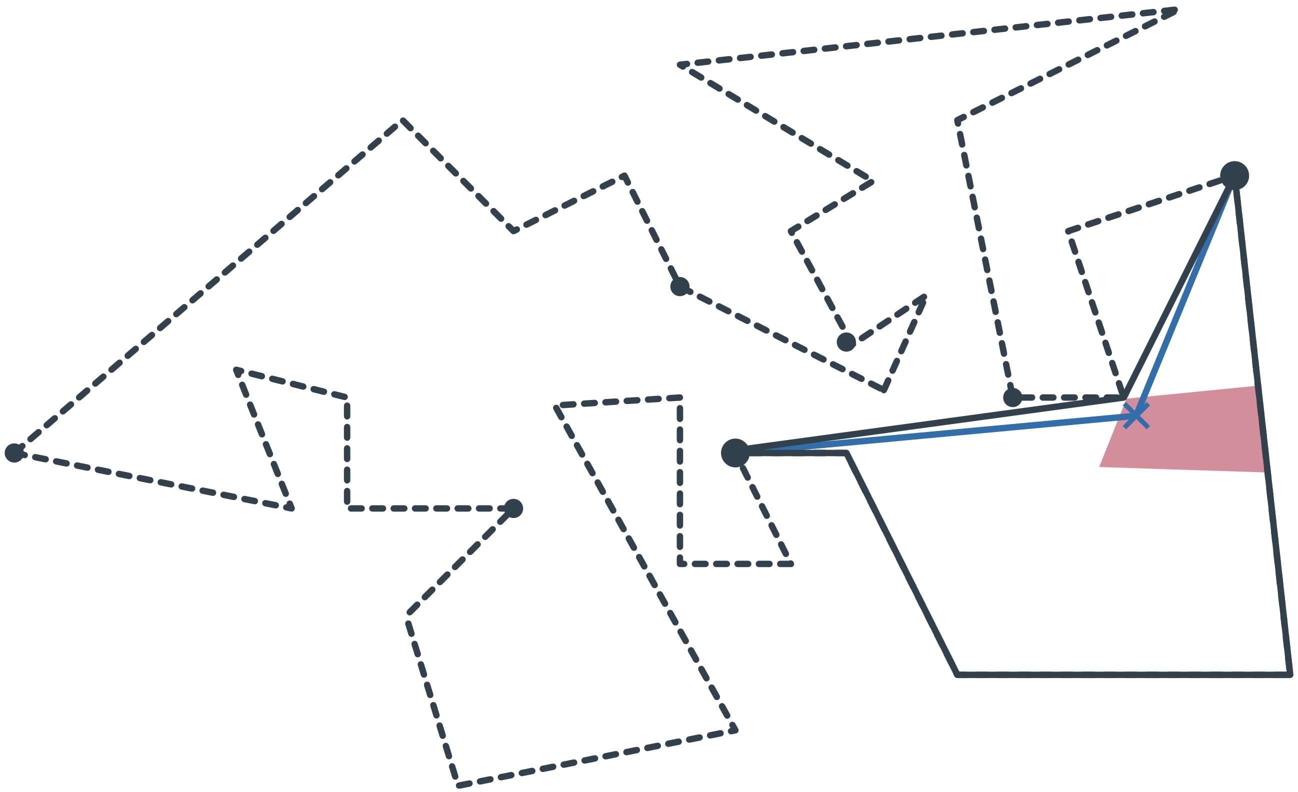

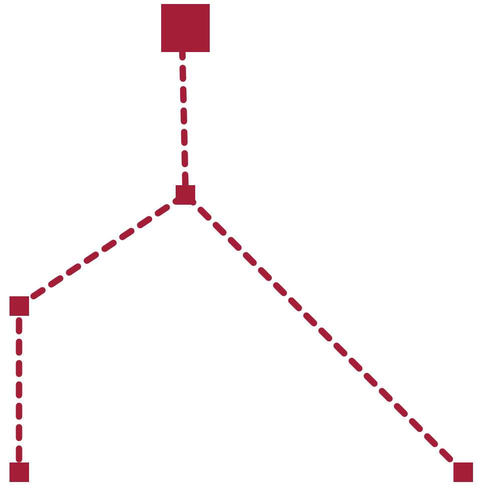

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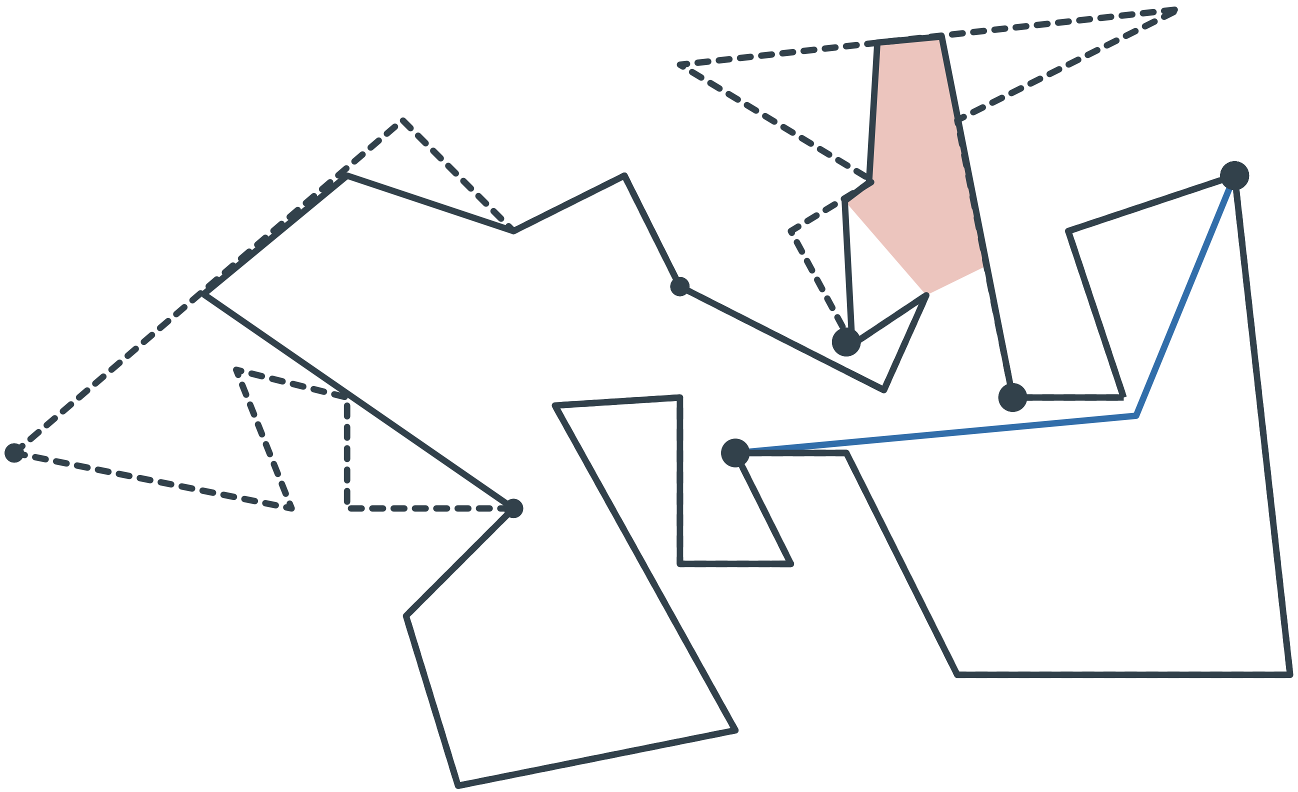

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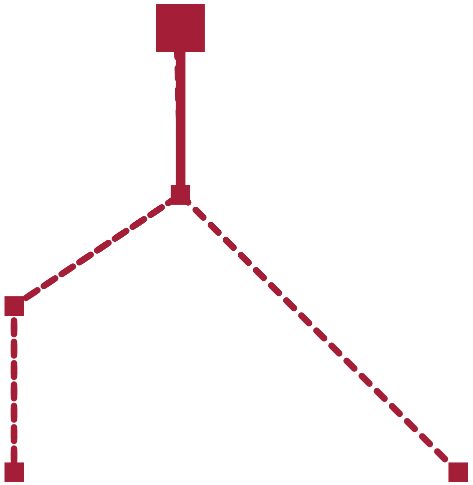

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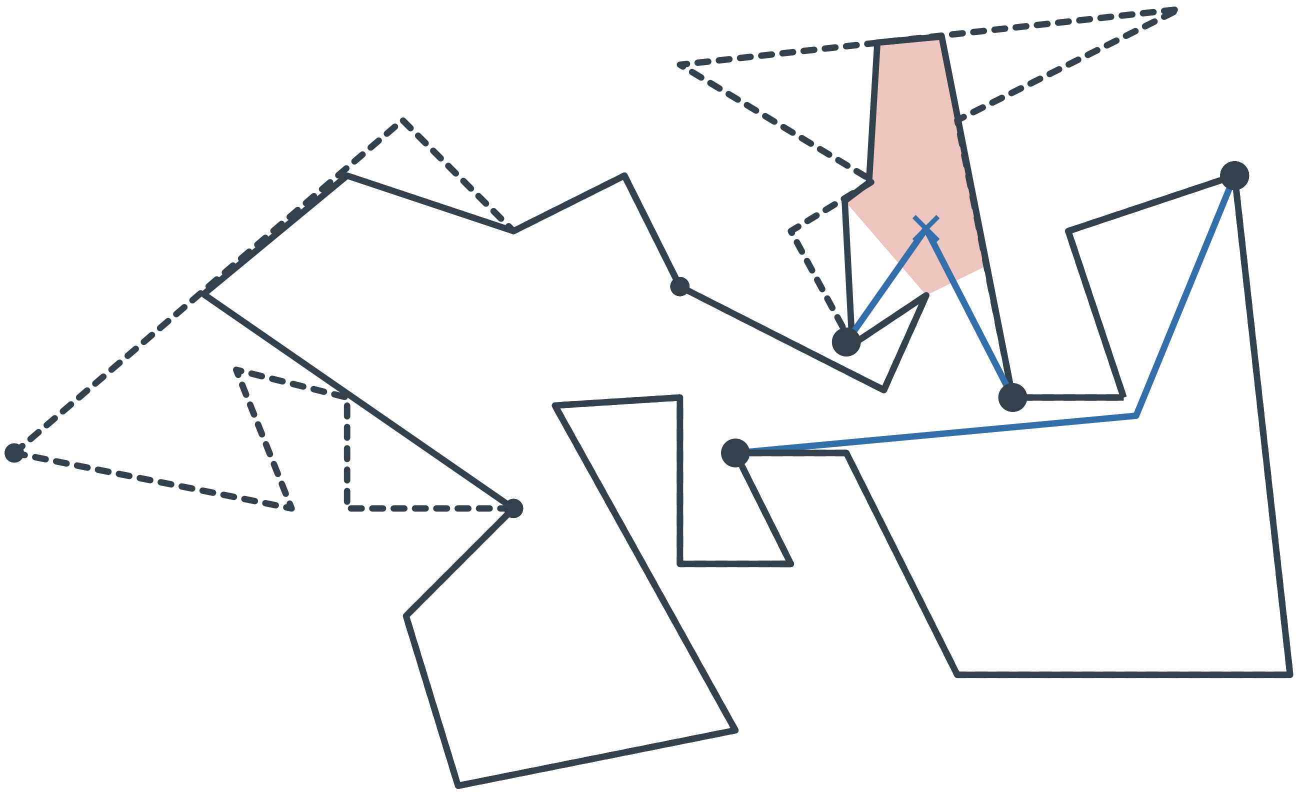

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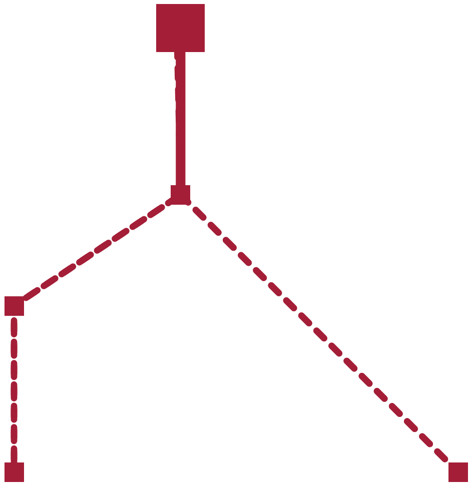

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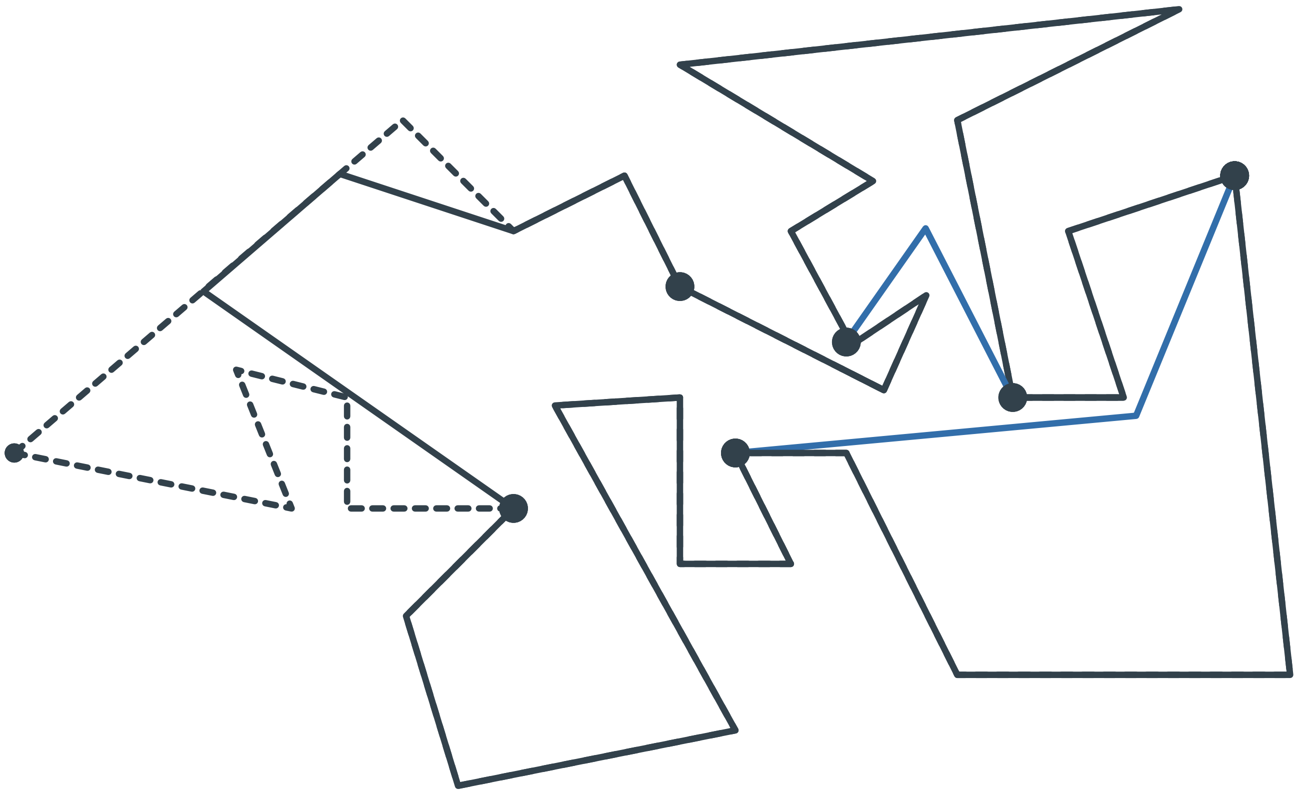

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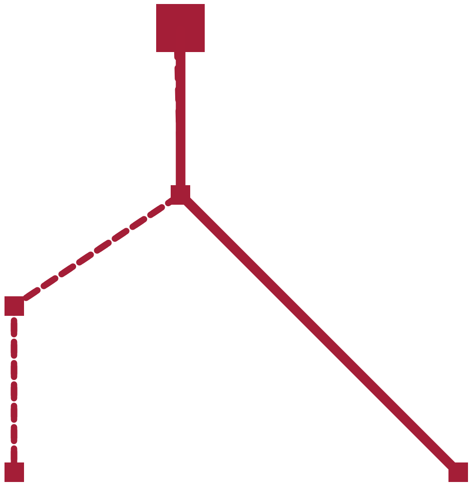

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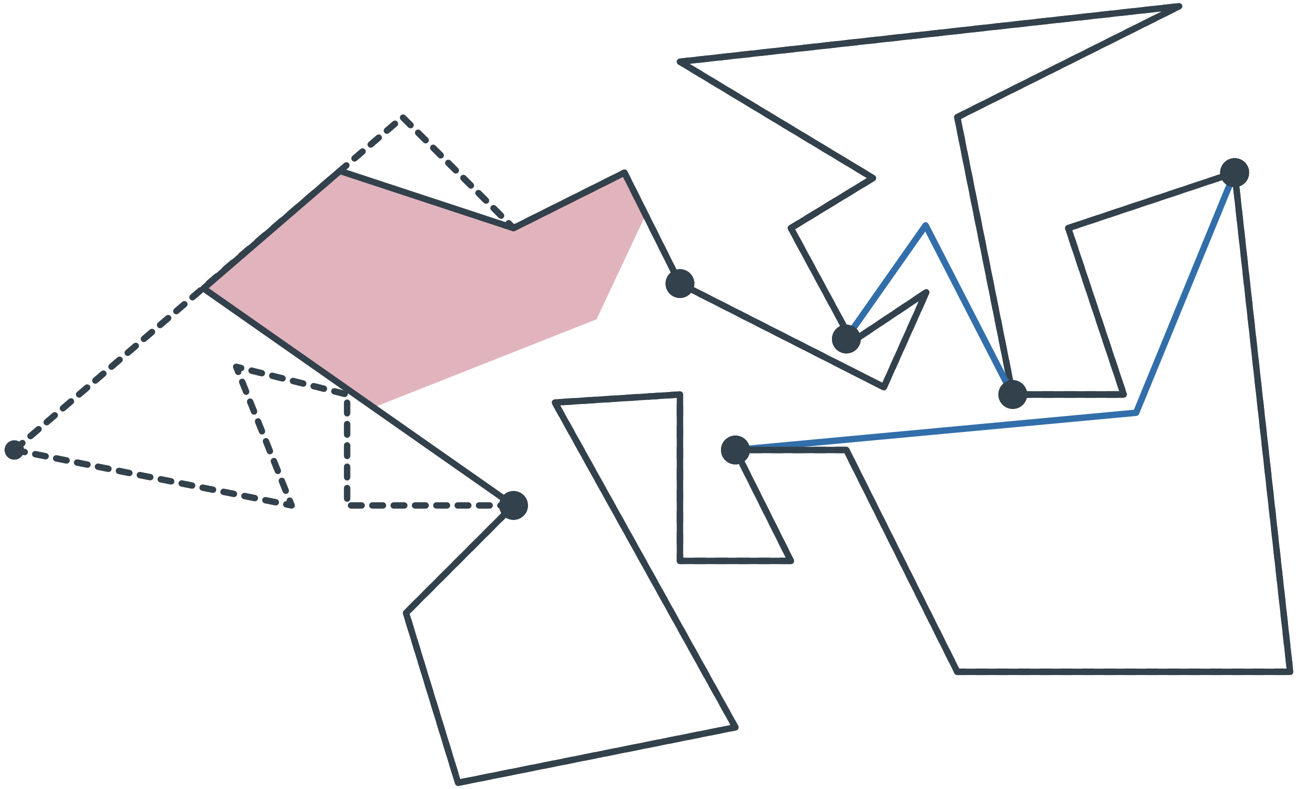

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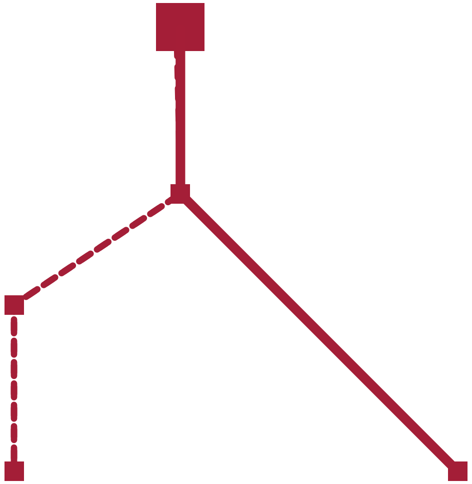

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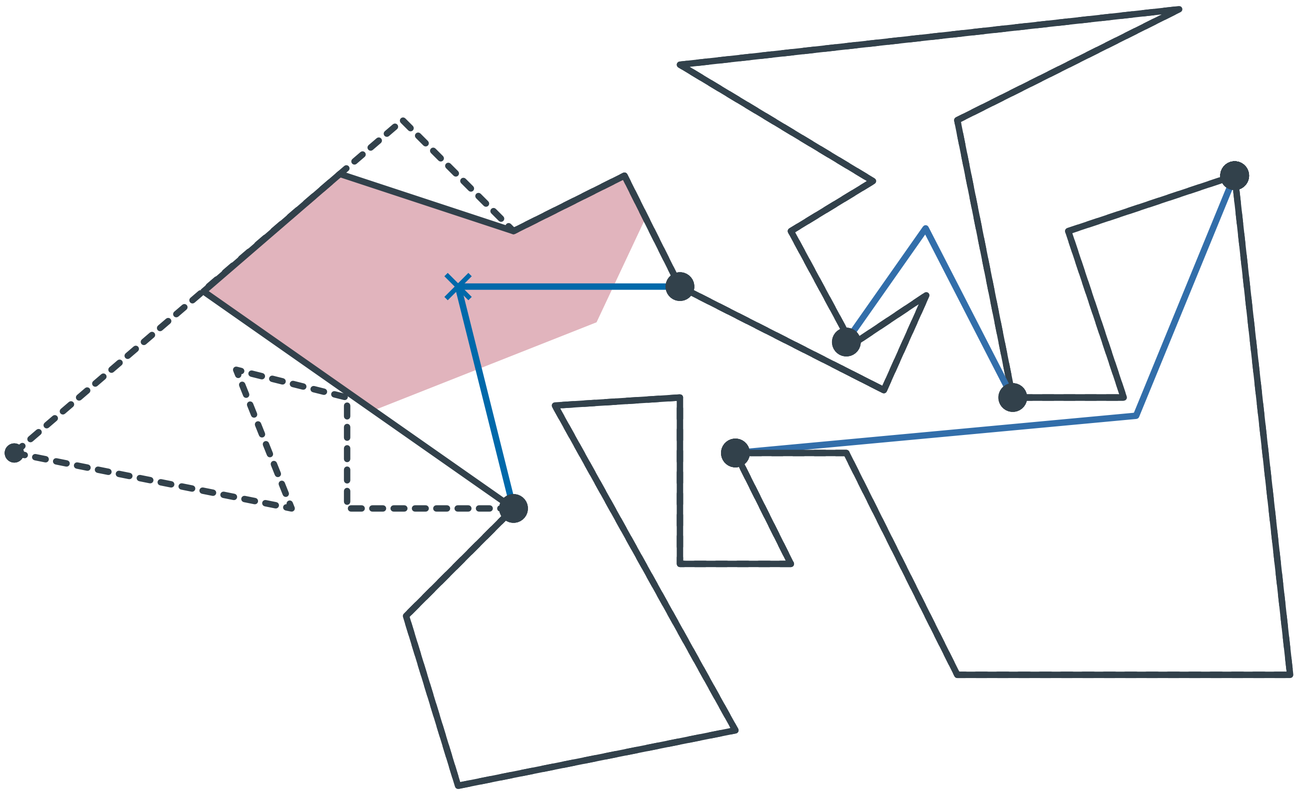

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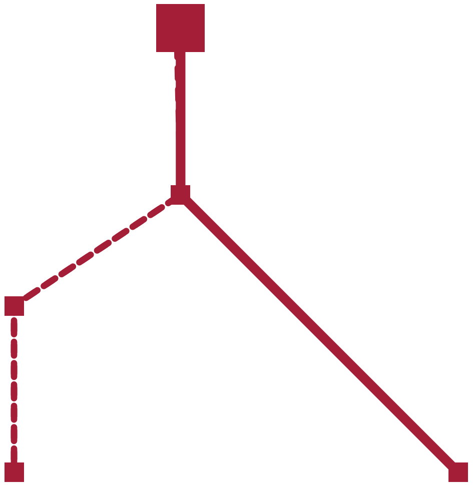

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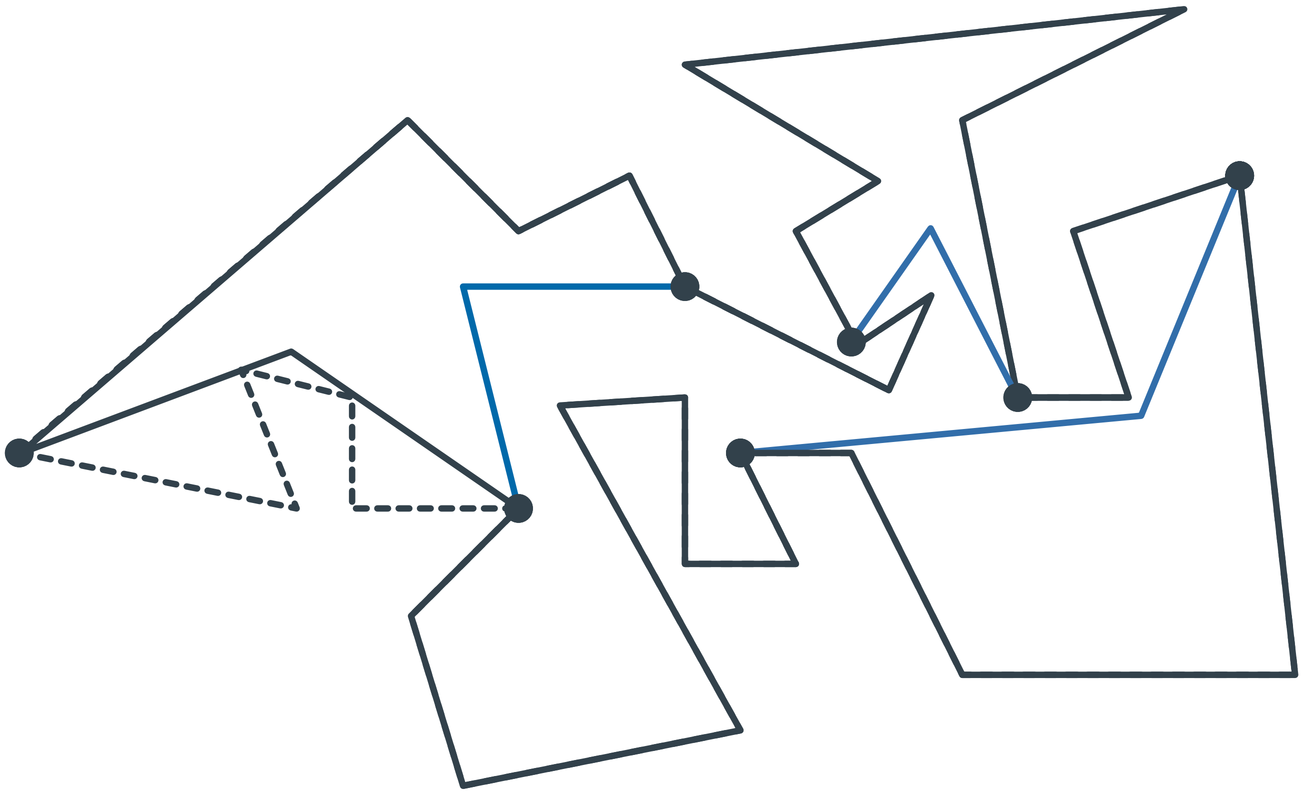

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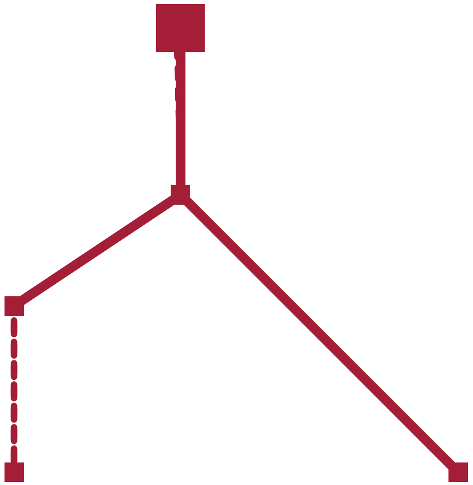

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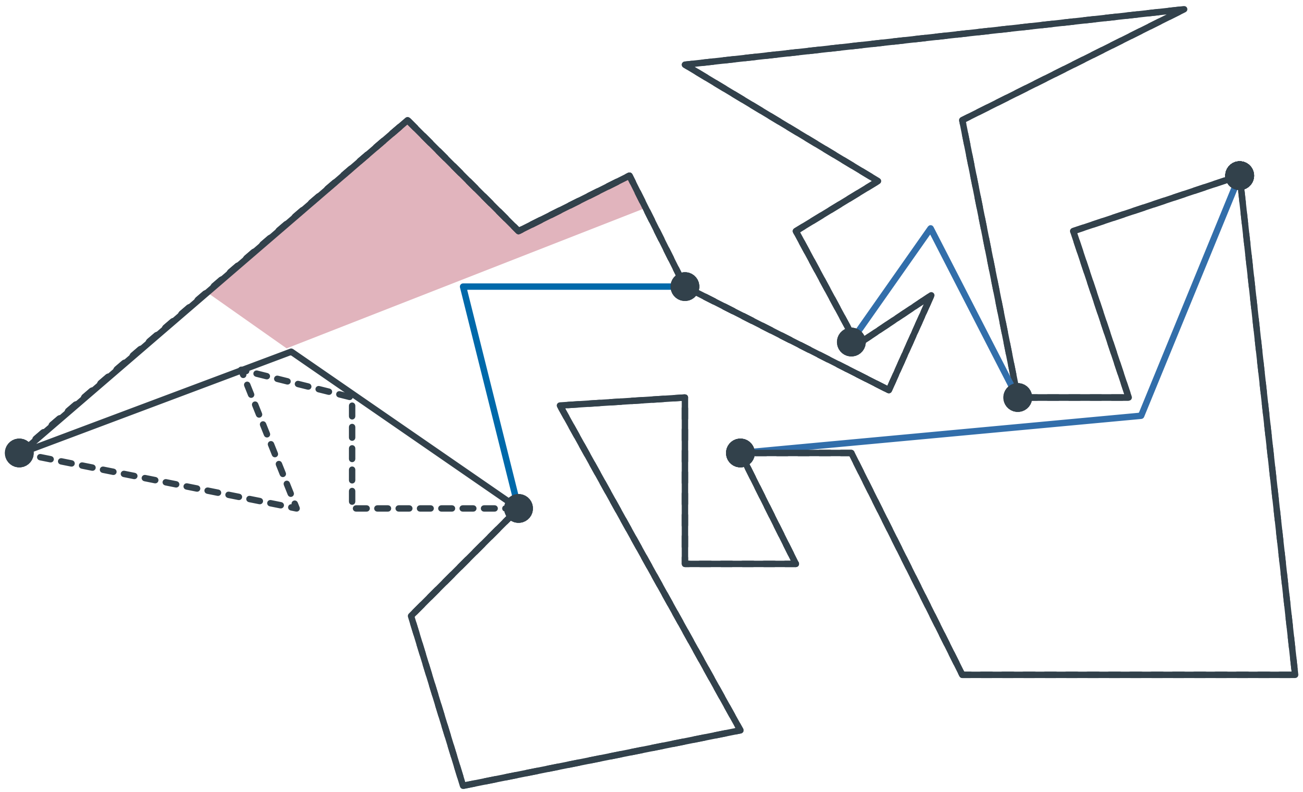

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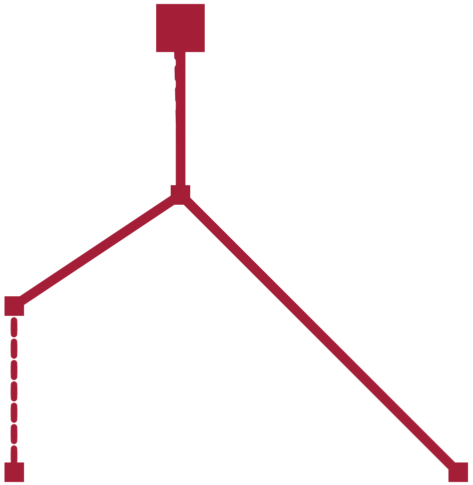

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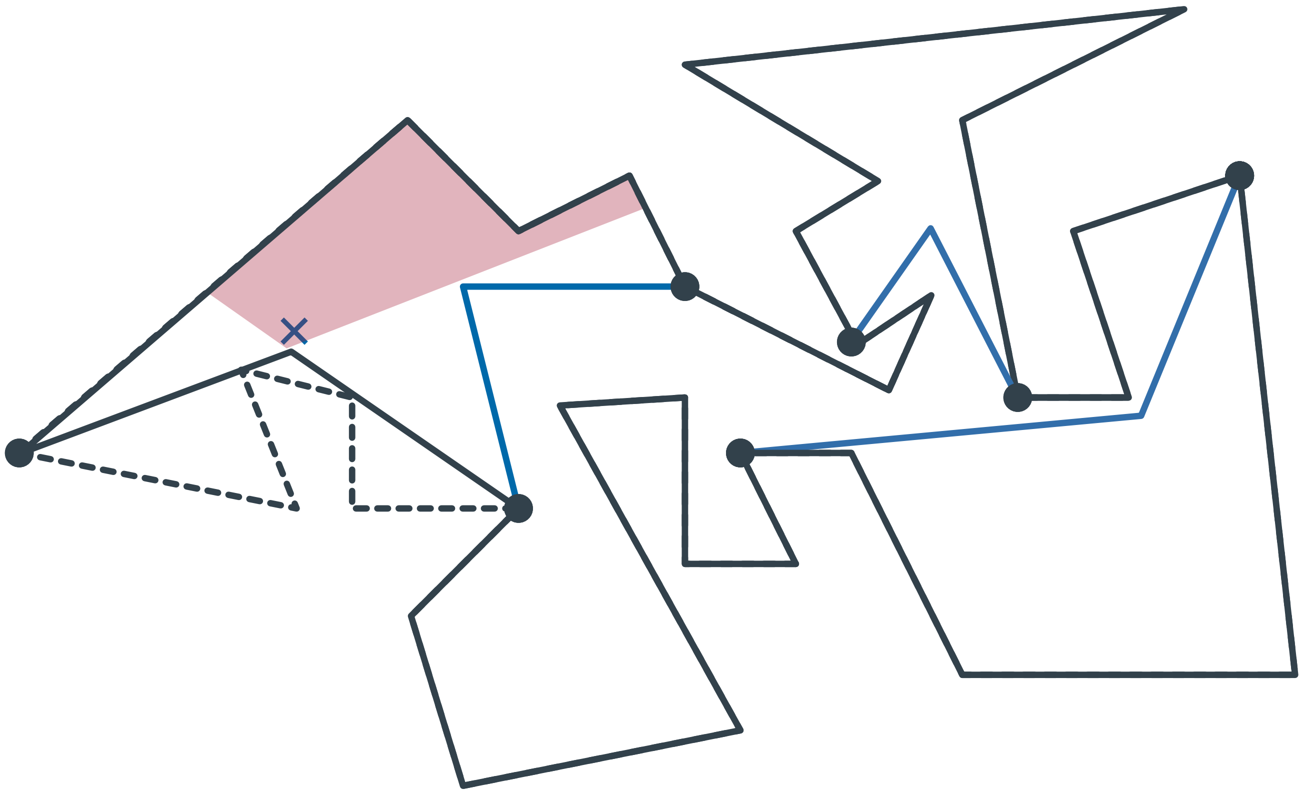

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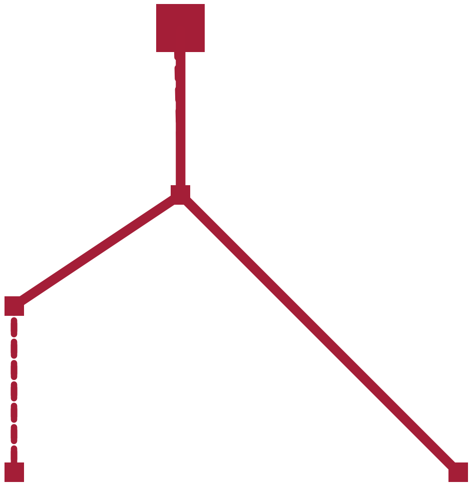

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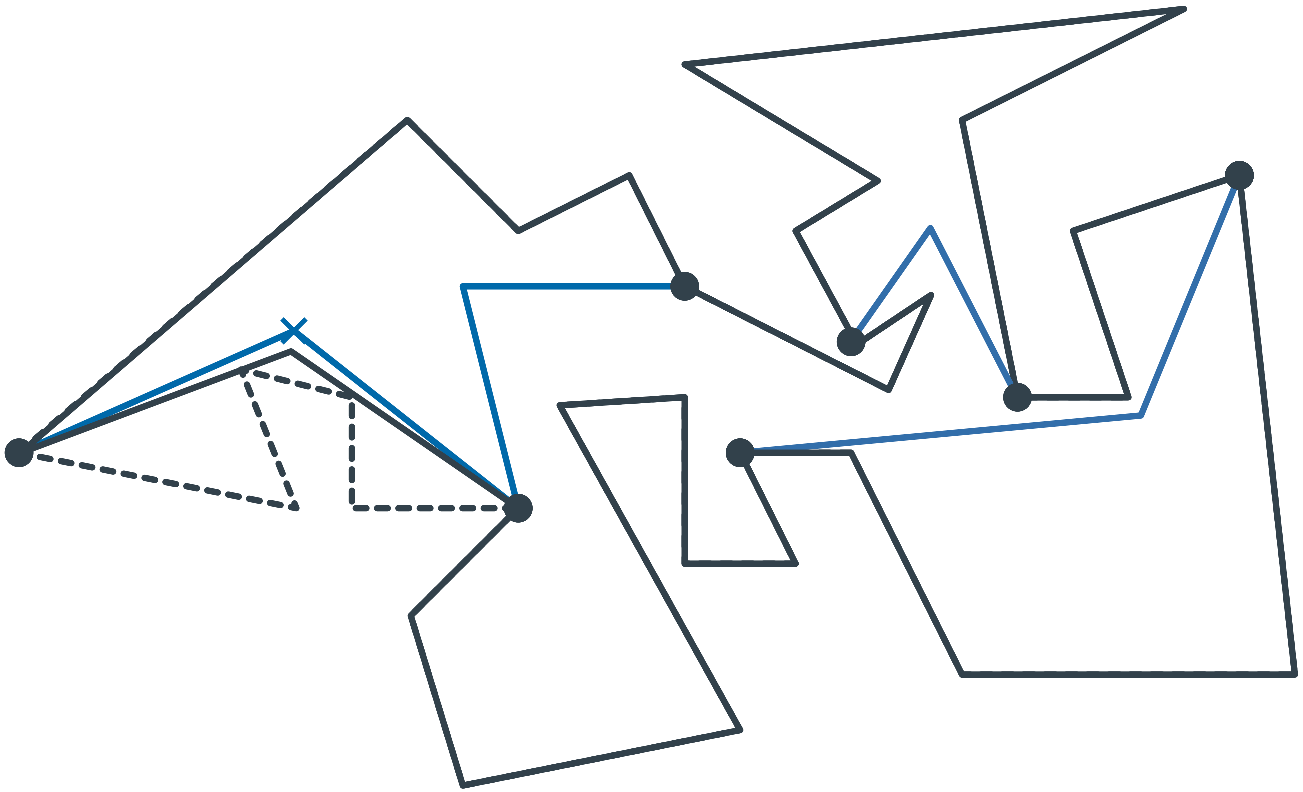

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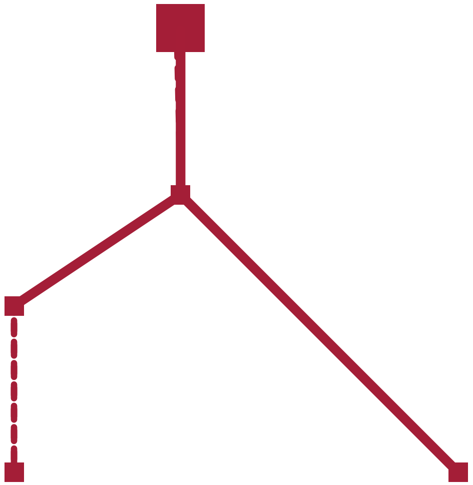

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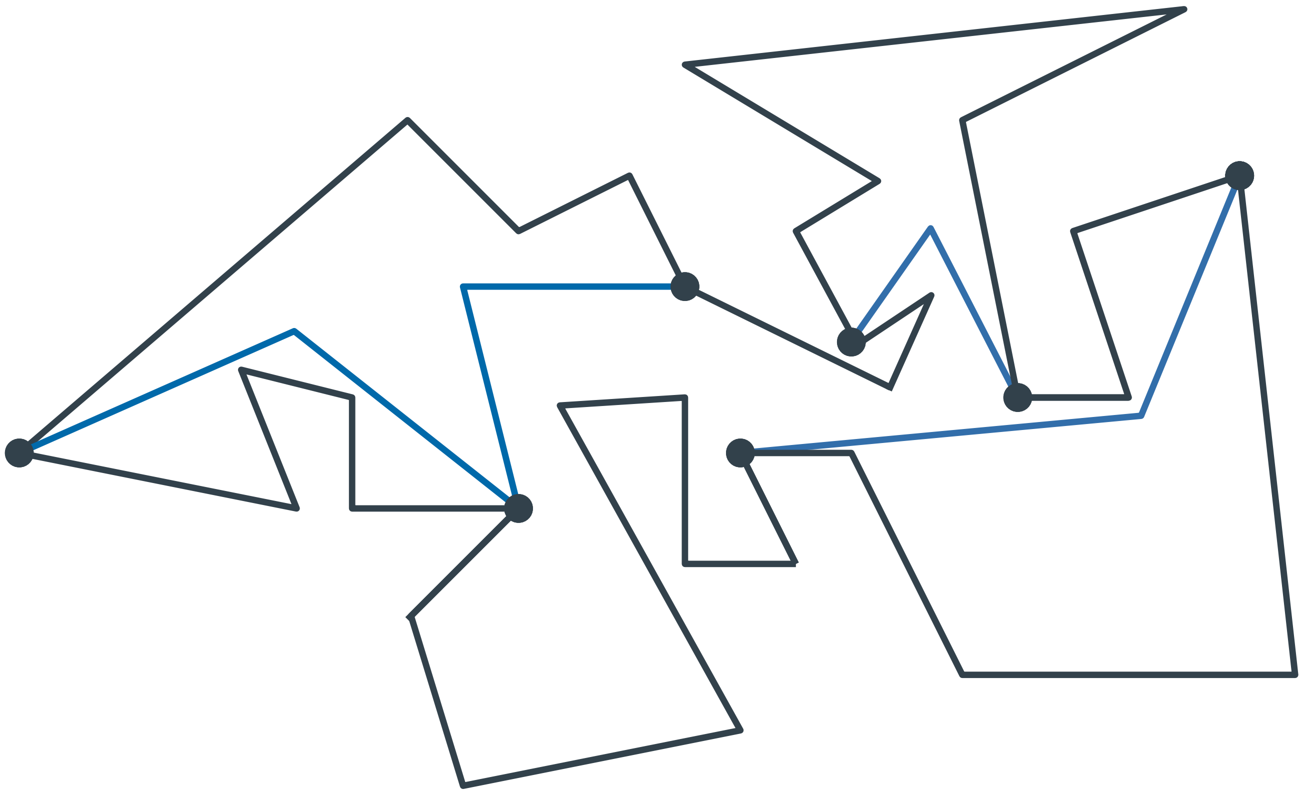

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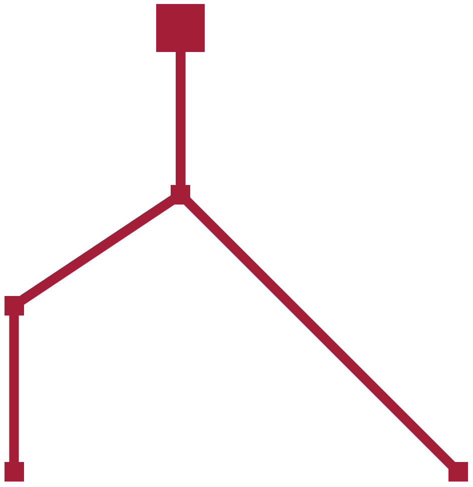

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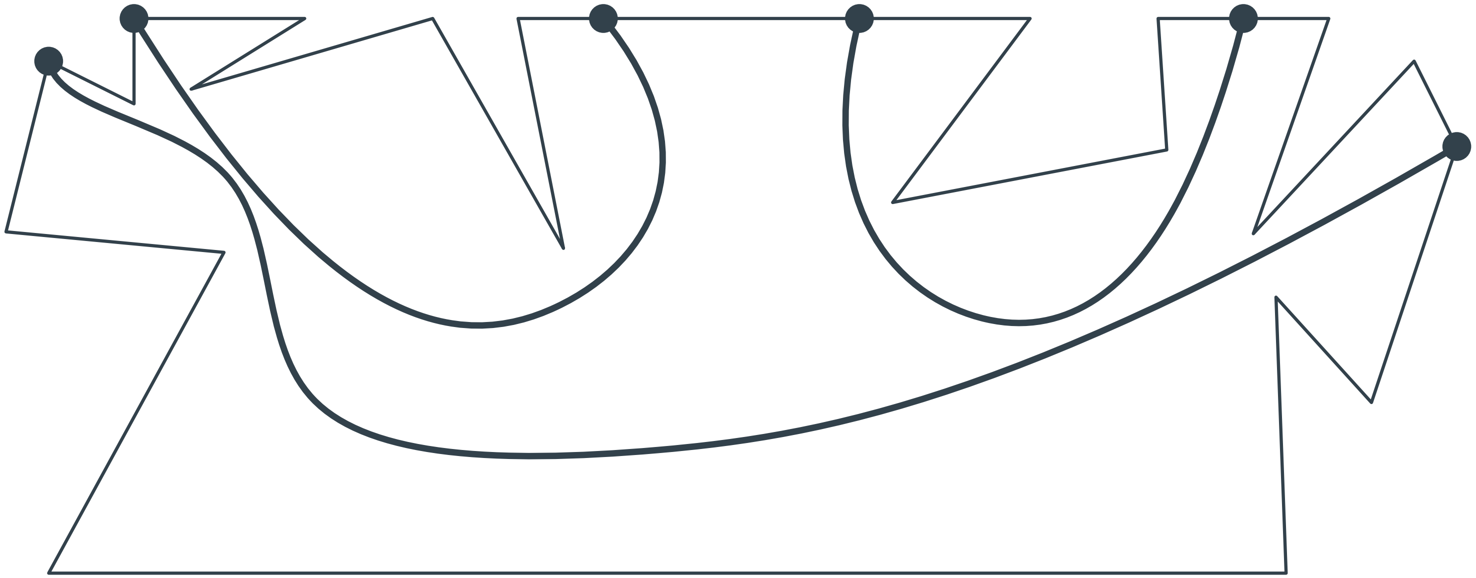

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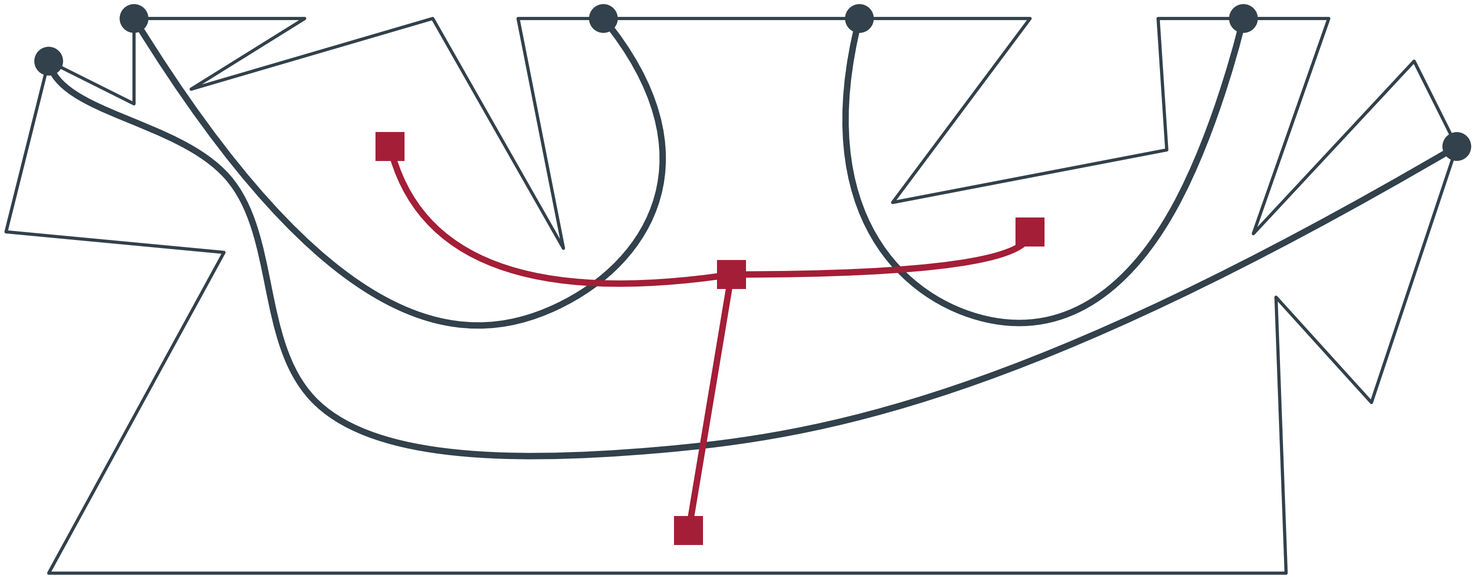

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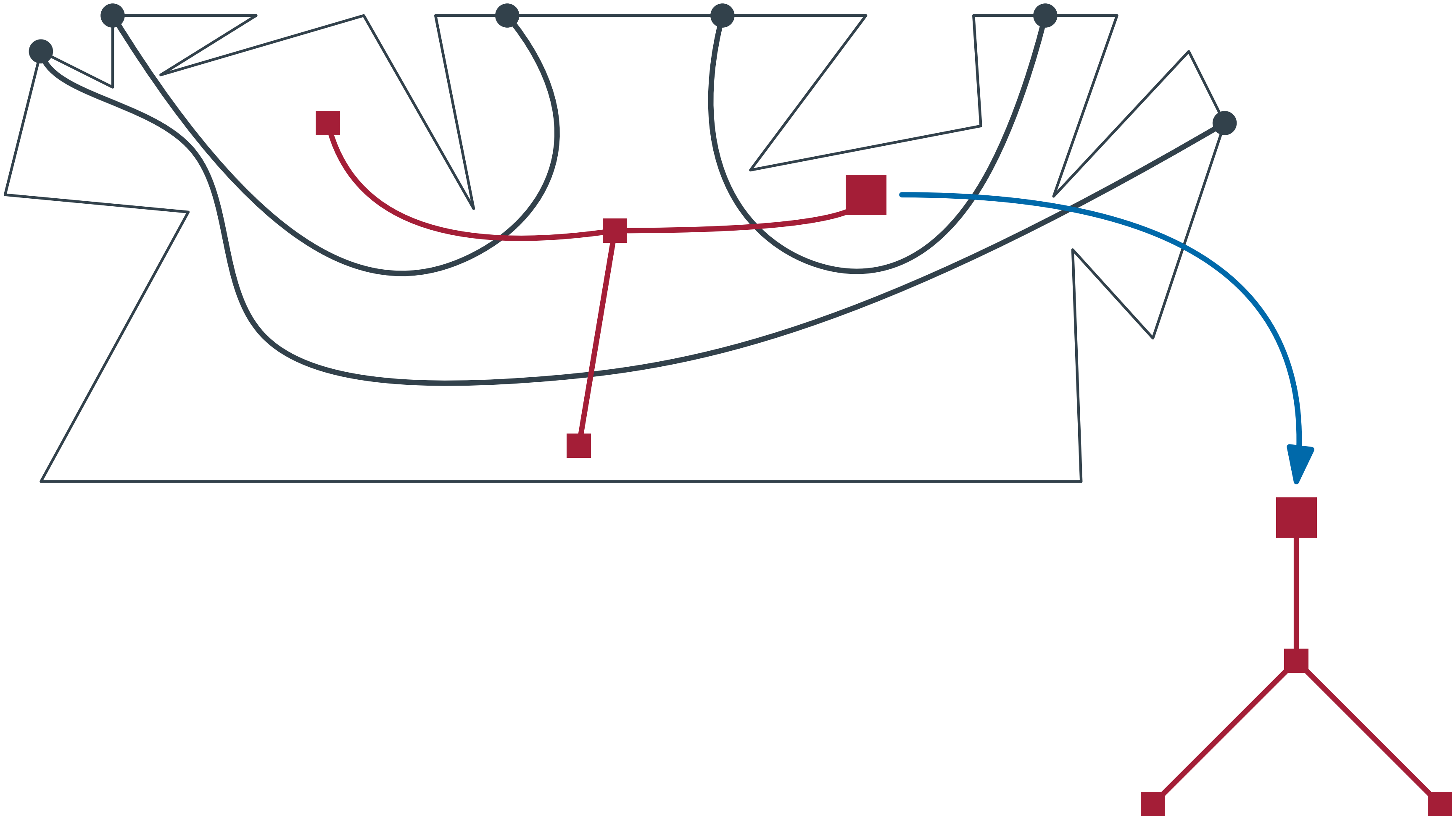

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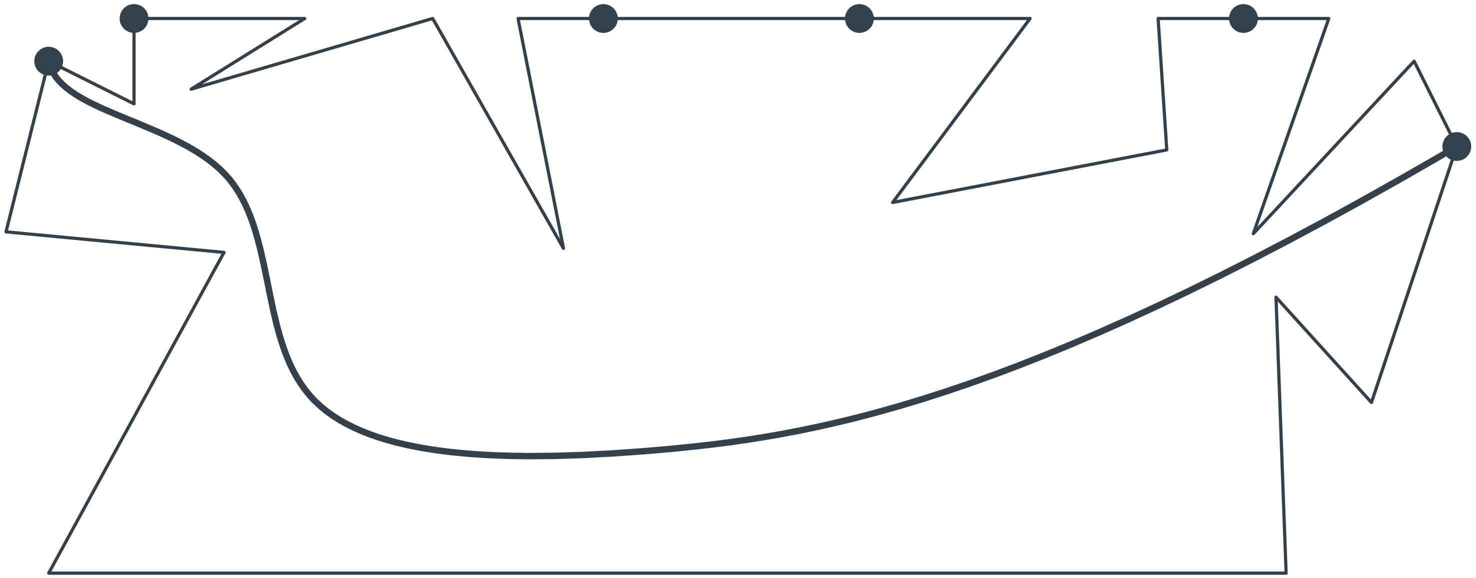

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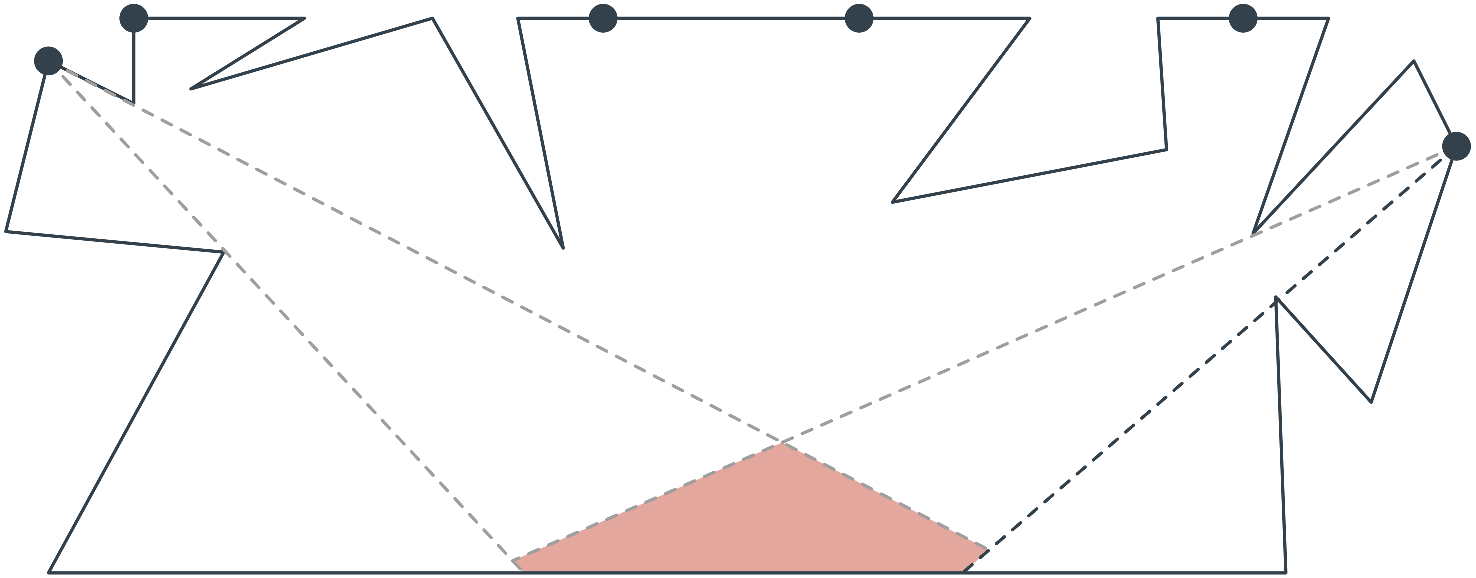

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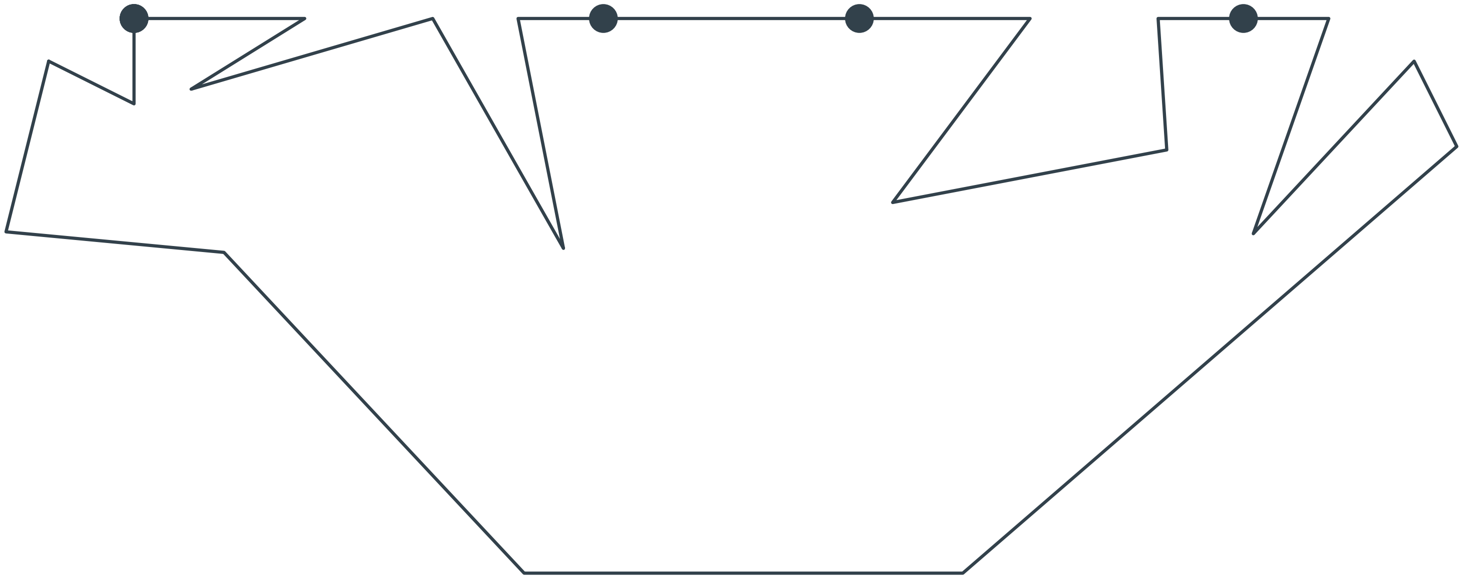

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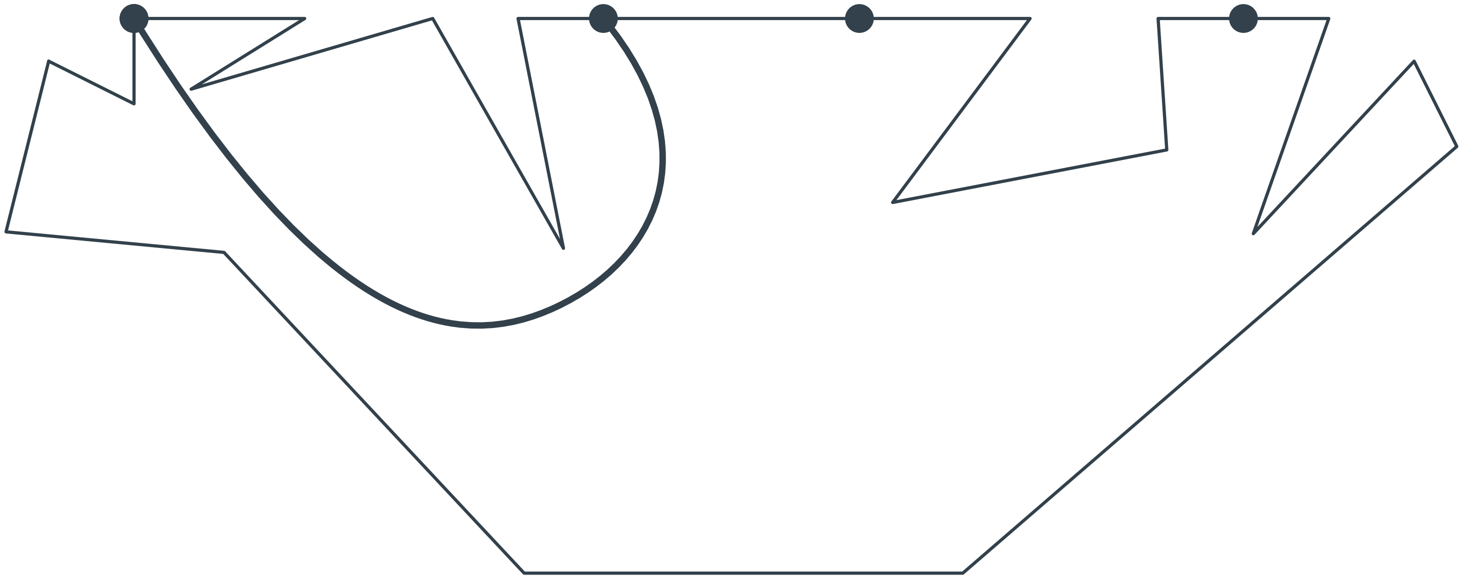

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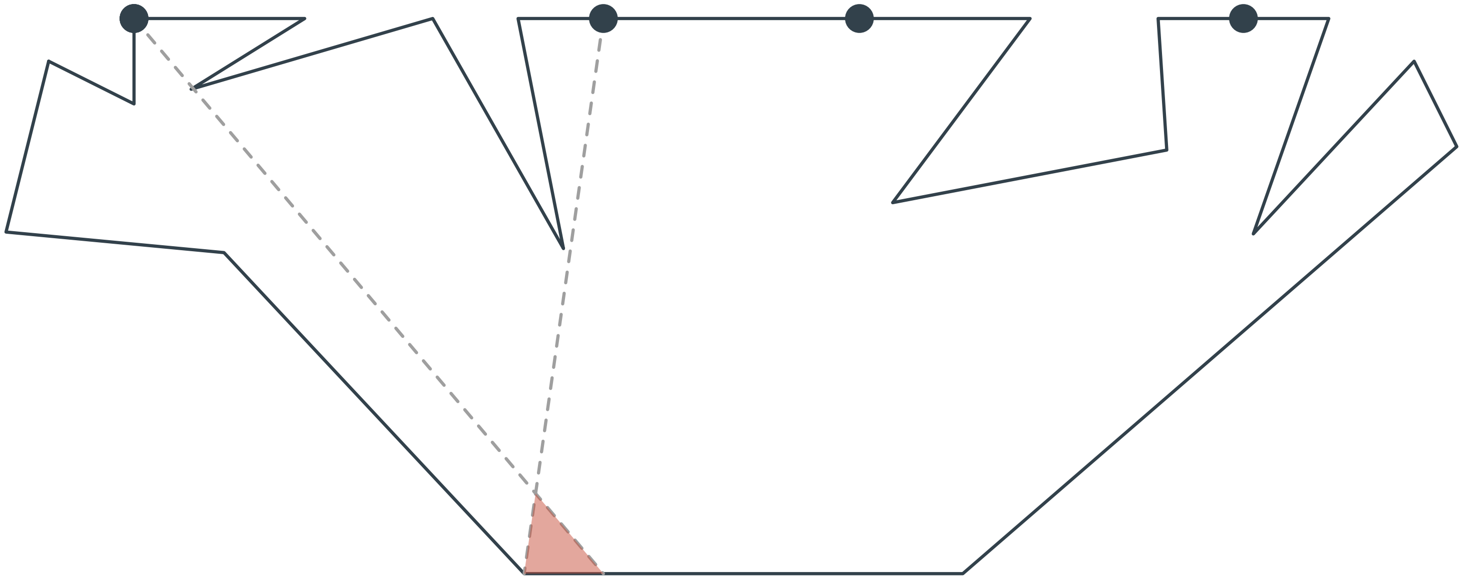

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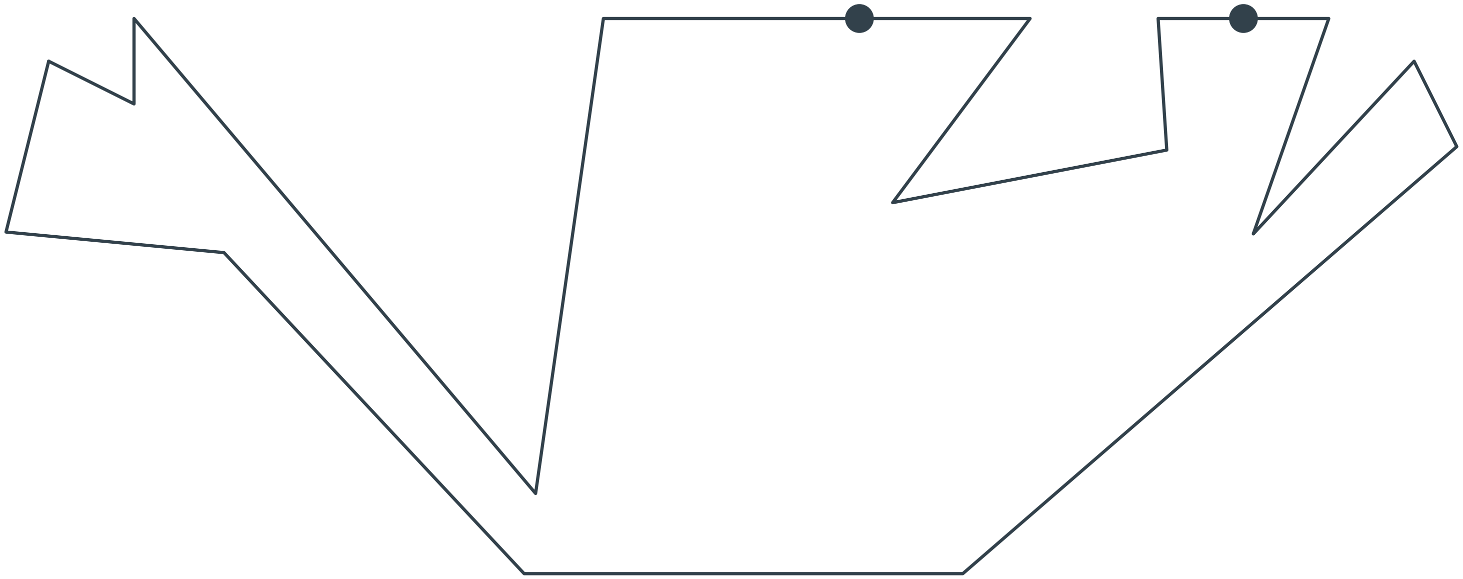

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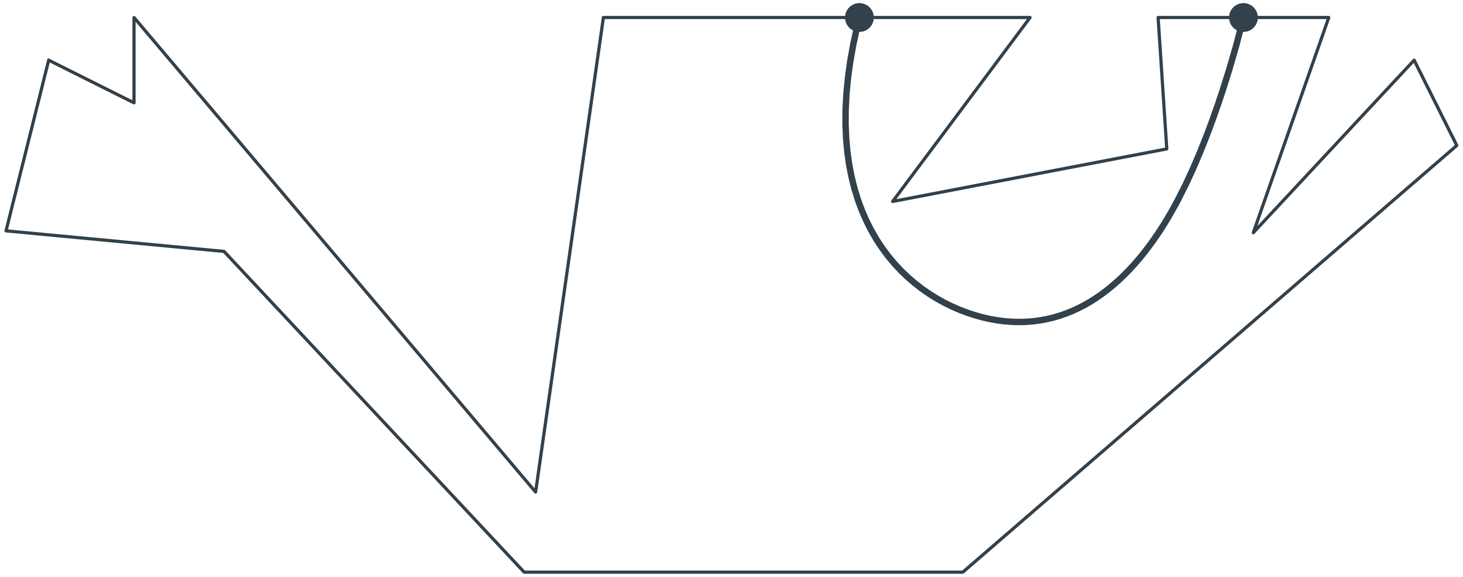

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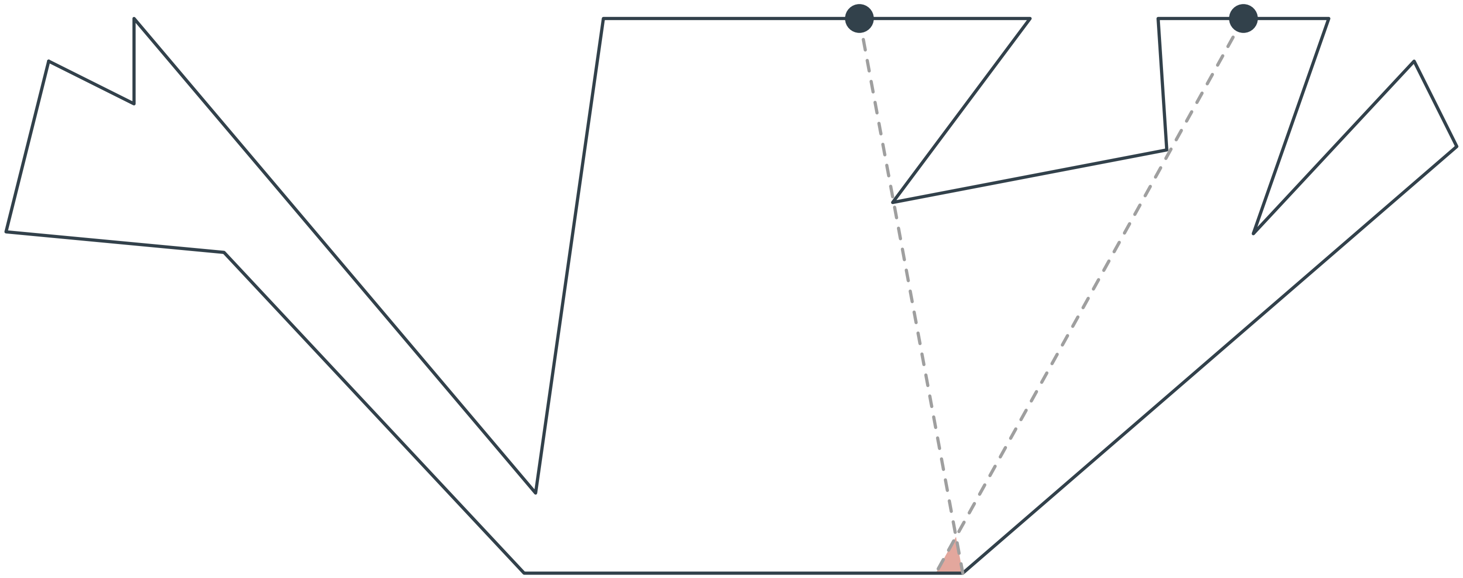

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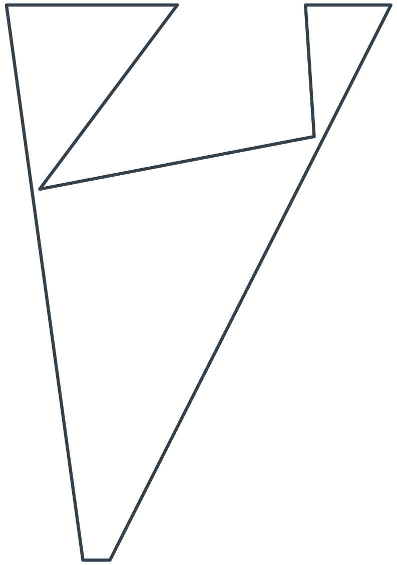

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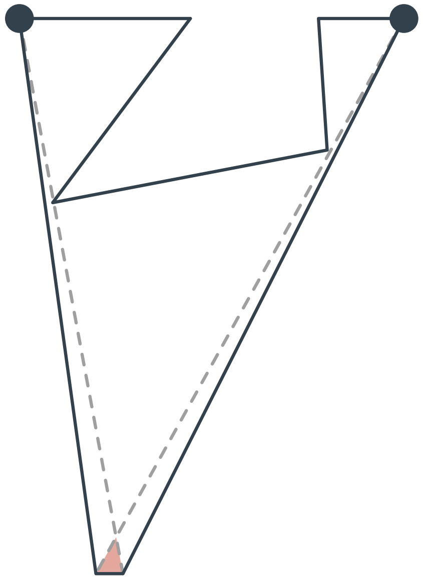

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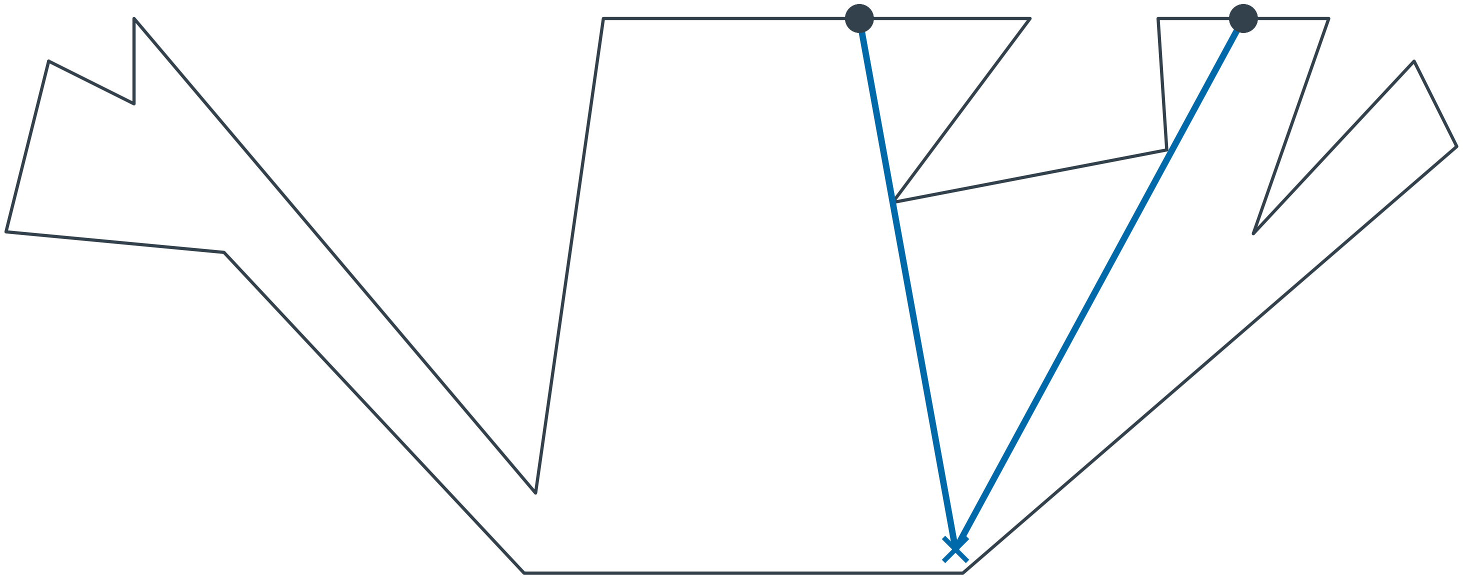

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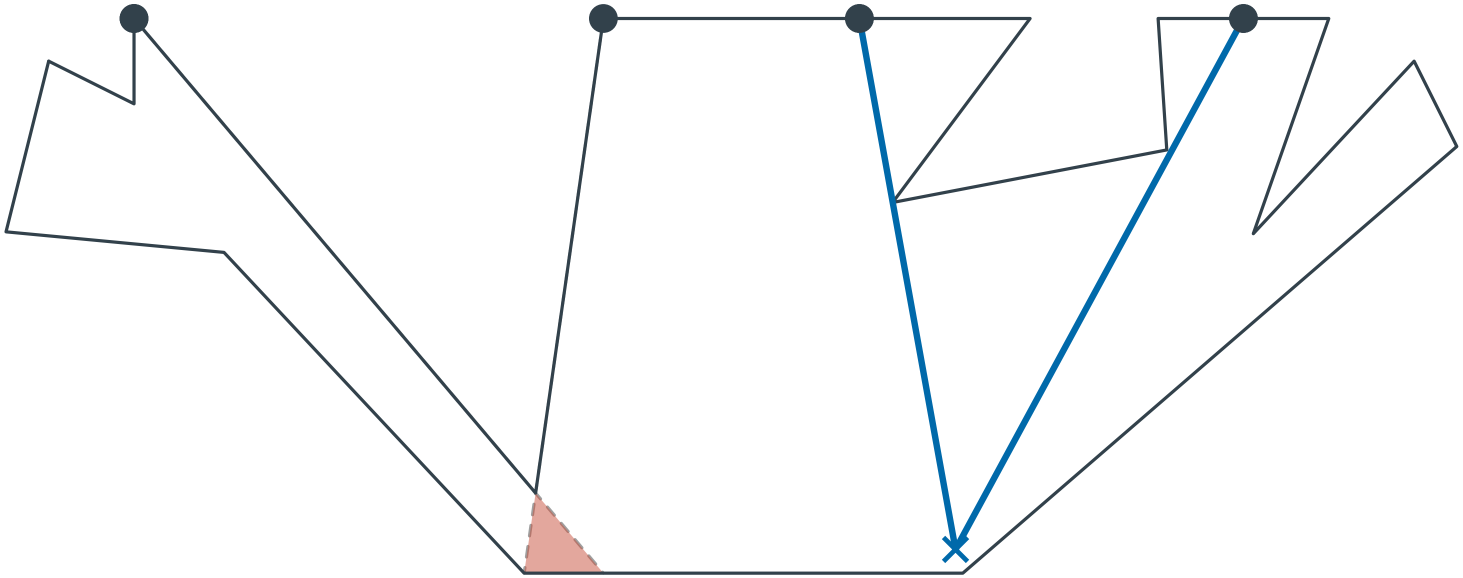
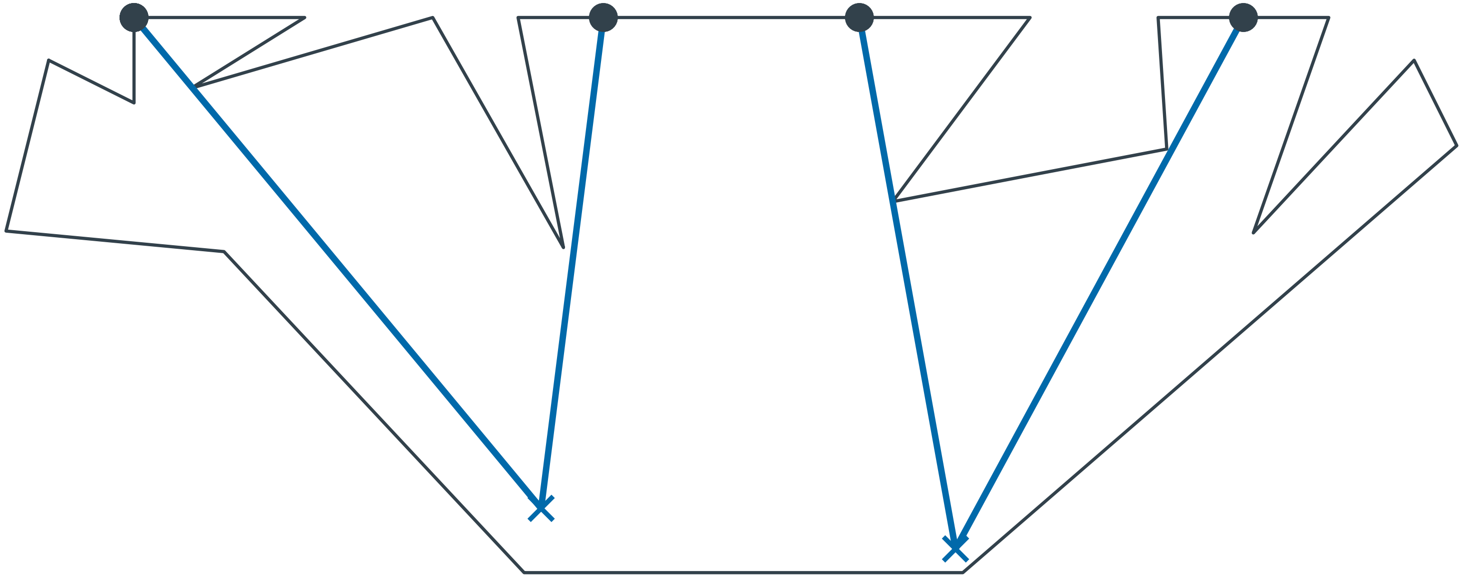
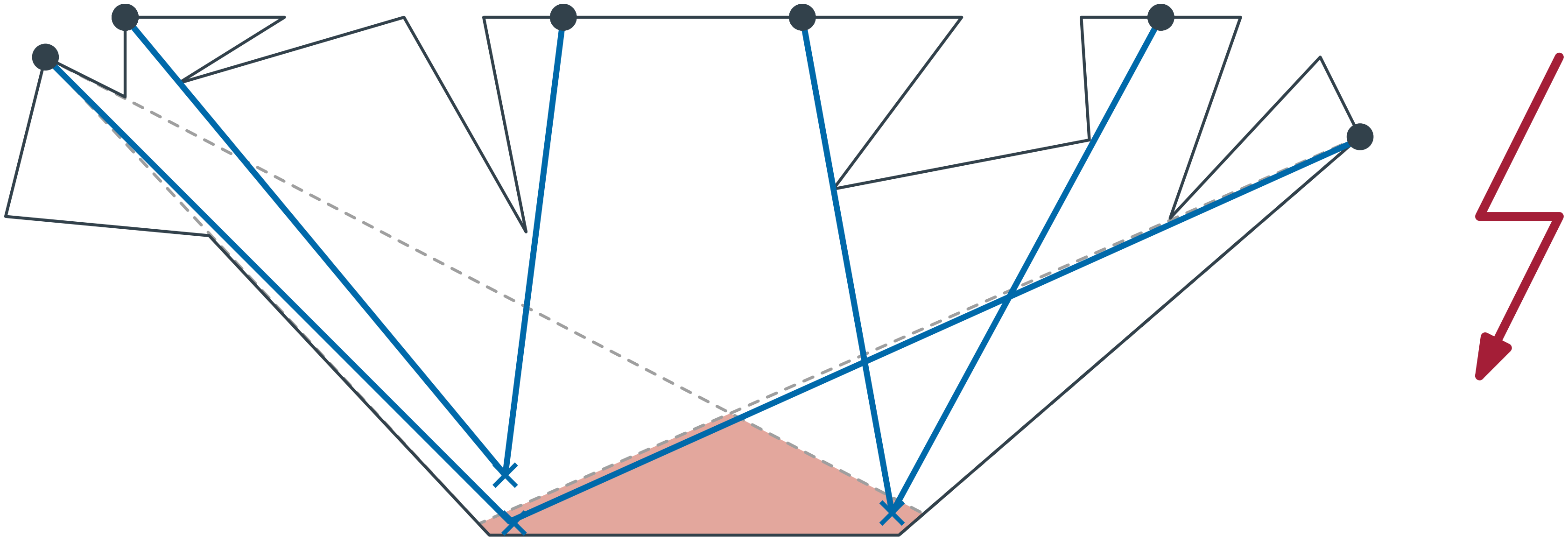


Why this order?



Why this order?



Running time

n:# vertices
m: # interior edges

- Tree has $O(m)$ nodes
- Per node: computer visibility polygon $O(n)$ (each refined polygon has $O(n)$ edges)

$\Longrightarrow$ $O(mn)$ time to decide whether G admits an outerplanar drawing with at ≤ 1 bend per edge that contains the given drawing of the outer face

In the positive case, the drawing can also be computed

Open problems

- More bends?
- Outer 1-planar drawings?